8.5.2 Lagrange Multipliers

The method of Lagrange multipliers is a strategy for finding the local maxima and minima of a function subject to equality constraints (i.e., subject to the condition that one or more equations have to be satisfied exactly by the chosen values of the variables). The basic idea is to convert a constrained problem into a form such that the derivative test of an unconstrained problem can still be applied.The Lagrange multiplier theorem roughly states that at any stationary point of the function that also satisfies the equality constraints, the gradient of the function at that point can be expressed as a linear combination of the gradients of the constraints at that point, with the Lagrange multipliers acting as coefficients. The great advantage of this method is that it allows the optimization to be solved without explicit parameterization in terms of the constraints. As a result, the method of Lagrange multipliers is widely used to solve challenging constrained optimization problems.

We want to optimize (i.e. find the minimum and maximum value of) a function, $f(x, y, z)$, subject to the constraint $g(x, y, z) = k$. Again, the constraint may be the equation that describes the boundary of a region or it may not be.

Method of Lagrange Multipliers

1. Solve the following system of equations.

$$\begin{aligned} \nabla f\,(x, y, z) &= \lambda\ \nabla g\,(x, y, z) \\ g\,(x, y, z) &= k \end{aligned}$$

2. Plug in all solutions, (x, y, z) , from the first step into $f(x, y, z)$ and identify the minimum and maximum values, provided they exist and $\nabla g \neq 0$ at the point. The constant,λ is called the Lagrange Multiplier.

Notice that the system of equations from the method actually has four equations, we just wrote the system in a simpler form. To see this let's take the first equation and put in the definition of the gradient vector to see what we get.

$$\langle f_x, f_y, f_z \rangle = \lambda \langle g_x, g_y, g_z \rangle = \langle \lambda g_x, \lambda g_y, \lambda g_z \rangle$$

In order for these two vectors to be equal the individual components must also be equal. So, we actually have three equations here.

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z$$

These three equations along with the constraint, $g(x, y, z) = c$, give four equations with four unknowns x, y, z, and λ . As a final note we also need to be careful with the fact that in some cases minimums and maximums won't exist even though the method will seem to imply that they do. In every problem we'll need to make sure that minimums and maximums will exist before we start the problem.

It is somewhat easier to understand two variable problems, so we begin with one as an example. Suppose the perimeter of a rectangle is to be 100 units.

Find the rectangle with largest area. This is a fairly straightforward problem from single variable calculus. We write down the two equations: $A = xy$, $P = 100 = 2x + 2y$, solve the second of these for y (or x), substitute into the first, and end up with a one-variable maximization problem. Let's now think of it differently: the equation $A = xy$ defines a surface, and the equation $100 = 2x + 2y$ defines a curve (a line, in this case) in the $x - y$ plane.

The gradient of $2x + 2y$ is $\langle 2, 2\rangle$, and the gradient of xy is $\langle y, x\rangle$. They are parallel when $\langle 2, 2\rangle = \lambda \langle y, x\rangle$, that is, when $2 = \lambda y$ and $2 = \lambda x$. We have two equations in three unknowns, which typically results in many solutions (as we expected). A third equation will reduce the number of solutions; the third equation is the original constraint, $100 = 2x + 2y$. So we have the following system to solve:

$$\begin{aligned} 2 &= \lambda y \\ 2 &= \lambda x \\ 100 &= 2x + 2y \end{aligned}$$

In the first two equations, λ can't be0, so we may divide by it to get $x = y = \frac{2}{\lambda}$. Substituting into the third equation we get

$$\begin{aligned} 2\frac{2}{\lambda} + 2\frac{2}{\lambda} &= 100 \\ \frac{8}{100} &= \lambda \end{aligned}$$

so $x = y = 25$.

Example 154 *Find the dimensions of the box with largest volume if the total surface area is 64 cm^2.*

Solution 155 *We first need to identify the function that we're going to optimize as well as the constraint. Let's set the length of the box to be x , the width of the box to be y and the height of the box to be z . Let's also note that because we're dealing with the dimensions of a box it is safe to assume that x, y , and z are all positive quantities. We want to find the largest volume and so the function that we want to optimize is given by,*

$$f(x, y, z) = xyz$$

Next, we know that the surface area of the box must be a constant 64. So this is the constraint. The surface area of a box is simply the sum of the areas of each of the sides so the constraint is given by,

$$2xy + 2xz + 2yz = 64 \Rightarrow xy + xz + yz = 32$$

Note that we divided the constraint by 2 to simplify the equation a little. Also, we get the function $g(x, y, z)$ from this.

$$g(x, y, z) = xy + xz + yz$$

The function itself, $f(x, y, z) = xyz$ will clearly have neither minimums or maximums unless we put some restrictions on the variables. The only real restriction that we've got is that all the variables must be positive. This, of course, instantly means that the function does have a minimum, zero, even though this is a silly value as it also means we pretty much don't have a box. It does however mean that we know the minimum of $f(x, y, z)$ does exist.
So, let's now see if $f(x, y, z)$ will have a maximum. Clearly, hopefully,$f(x, y, z)$ will not have a maximum if all the variables are allowed to increase without bound. That however, can't happen because of the constraint,

$$xy + xz + yz = 32$$

Here we've got the sum of three positive numbers (remember that we x , y , and z are positive because we are working with a box) and the sum must equal 32. So, if one of the variables gets very large, say x , then because each of the products must be less than 32 both y and z must be very small to make sure the first two terms are less than 32. So, there is no way for all the variables to increase without bound and so it should make some sense that the function, $f(x, y, z) = xyz$, will have a maximum. This is not an exact proof that $f(x, y, z)$ will have a maximum but it should help to visualize that $f(x, y, z)$ should have a maximum value as long as it is subject to the constraint. Here are the four equations that we need to solve.

$$yz = \lambda(y + z)(fx = \lambda gx) \tag{1}$$

$$xz = \lambda(x + z)(fy = \lambda gy) \tag{2}$$

$$xy = \lambda(x + y)(fz = \lambda gz) \tag{3}$$

$$xy + xz + yz = 32(g(x, y, z) = 32) \tag{4}$$

There are many ways to solve this system. We'll solve it in the following way. Let's multiply equation (1) by x , equation (2 by y and equation (3) by z . This gives,

$$xyz = \lambda x(y + z) \tag{5}$$

$$xyz = \lambda y(x + z \tag{6}$$

$$xyz = \lambda z(x + y) \tag{7}$$

Now notice that we can set equations (5) and (6) equal. Doing this gives,

$$\lambda x(y + z) = \lambda y(x + z)$$

$$\lambda(xy + xz) - \lambda(yx + yz) = 0$$

$$\lambda(xz - yz) = 0 \Rightarrow \lambda = 0 \ or \ xz = yz$$

This gave two possibilities. The first, $\lambda = 0$ is not possible since if this was the case equation (1) would reduce to

$$yz = 0 \Rightarrow y = 0 \ or \ z = 0$$

Since we are talking about the dimensions of a box neither of these are possible so we can discount $\lambda = 0$. This leaves the second possibility.

$$xz = yz$$

Since we know that $z \neq 0$ (again since we are talking about the dimensions of a box) we can cancel the z from both sides. This gives,

$$x = y \tag{8}$$

Next, let's set equations (6) and (7) equal. Doing this gives,

$$\lambda y(x + z) = \lambda z(x + y)$$

$$\lambda(yx + yz - zx - zy) = 0$$

$$\lambda(yx - zx) = 0 \Rightarrow \lambda = 0 \text{ or } yx = zx$$

As already discussed we know that $\lambda = 0$ won't work and so this leaves,

$$yx = zx$$

We can also say that $x \neq 0$ since we are dealing with the dimensions of a box so we must have,

$$z = y \tag{9}$$

Plugging equations (8) and (9) into equation (4) we get,

$$y^2 + y^2 + y^2 = 3y^2 = 32 \qquad y = \pm\sqrt{\frac{32}{3}} = \pm 3.266$$

However, we know that y must be positive since we are talking about the dimensions of a box. Therefore, the only solution that makes physical sense here is

$$x = y = z = 3.266$$

So, it looks like we've got a cube. We should be a little careful here. Since we've only got one solution we might be tempted to assume that these are the dimensions that will give the largest volume. Anytime we get a single solution we really need to verify that it is a maximum (or minimum if that is what we are looking for). This is actually pretty simple to do. First, let's note that the volume at our solution above is,

$$V = f(\sqrt{\frac{32}{3}}, \sqrt{\frac{32}{3}}, \sqrt{\frac{32}{3}}) = (\sqrt{\frac{32}{3}})^3 = 34.8376$$

Now, we know that a maximum of $f(x, y, z)$ will exist ("proved" that earlier in the solution) and so to verify that that this really is a maximum all we need to do if find another set of dimensions that satisfy our constraint and check the volume. If the volume of this new set of dimensions is smaller that the volume above then we know that our solution does give a maximum. If, on the other *hand, the new set of dimensions give a larger volume we have a problem. We only have a single solution and we know that a maximum exists and the method should generate that maximum. So, in this case, the likely issue is that we will have made a mistake somewhere and we'll need to go back and find it. So, let's find a new set of dimensions for the box. The only thing we need to worry about is that they will satisfy the constraint. Outside of that there aren't other constraints on the size of the dimensions. So, we can freely pick two values and then use the constraint to determine the third value. Let's choose* $x = y = 1$ *. No reason for these values other than they are "easy" to work with. Plugging these into the constraint gives,*

$$1 + z + z = 32 \rightarrow 2z = 31 \rightarrow z = \frac{31}{2}$$

So, this is a set of dimensions that satisfy the constraint and the volume for this set of dimensions is,

$$V = f(1, 1, \frac{31}{2}) = \frac{31}{2} = 15.5 < 34.8376$$

So, the new dimensions give a smaller volume and so our solution above is, in fact, the dimensions that will give a maximum volume of the box are

$$x = y = z = 3.266$$