8.6 Vector Fields

Definition 156 *A vector field on two (or three) dimensional space is a function* $\overrightarrow{F}$ *that assigns to each point* (x, y) *(or* (x, y, z)*) a two (or three dimensional) vector given by* $\overrightarrow{F}(x, y)(or \overrightarrow{F}(x, y, z))$.

That may not make a lot of sense, but most people do know what a vector field is, or at least they've seen a sketch of a vector field. If you've seen a current sketch giving the direction and magnitude of a flow of a fluid or the direction and magnitude of the winds then you've seen a sketch of a vector field.

The standard notation for the function $\overrightarrow{F}$

$$\vec{F}(x, y) = P(x, y)\,\vec{i} + Q(x, y)\,\vec{j}$$
$$\vec{F}(x, y, z) = P(x, y, z)\,\vec{i} + Q(x, y, z)\,\vec{j} + R(x, y, z)\,\vec{k}$$

depending on whether or not we're in two or three dimensions. The functionP, Q, R (if it is present) are sometimes called scalar functions.

Example 157 *Sketch each of the following vector fields.*

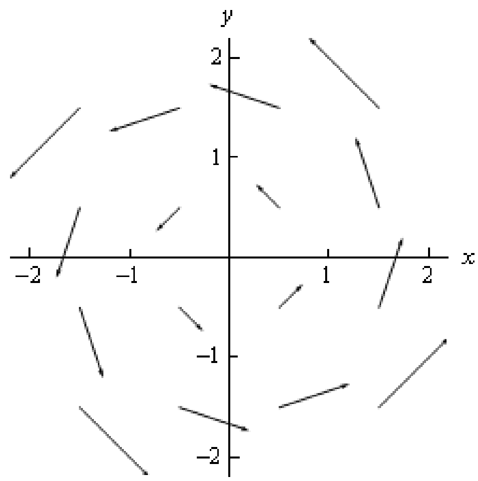
a $F(x, y) = -yi + xj$

b $F(x, y, z) = 2xi - 2yj - 2xk$

Solution 158 *To graph the vector field we need to get some "values" of the function. This means plugging in some points into the function. Here are a couple of evaluations.*

$$\vec{F}\left(\frac{1}{2}, \frac{1}{2}\right) = -\frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}$$
$$\vec{F}\left(\frac{1}{2}, -\frac{1}{2}\right) = -\left(-\frac{1}{2}\right)\vec{i} + \frac{1}{2}\vec{j} = \frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}$$
$$\vec{F}\left(\frac{3}{2}, \frac{1}{4}\right) = -\frac{1}{4}\vec{i} + \frac{3}{2}\vec{j}$$

So, just what do these evaluations tell us? Well the first one tells us that at the point $(1/2, 1/2)$ *we will plot the vector* $-1/2i + 1/2j$ *. Likewise, the third evaluation tells us that at the point* $(3/2, 1/4)$ *we will plot the vector* $-1/4i+3/2j$ *. We can continue in this fashion plotting vectors for several points and we'll get the following sketch of the vector field.*



Now that we've seen a couple of vector fields let's notice that we've already seen a vector field function. In the second chapter we looked at the gradient vector. Recall that given a function $f(x, y, z)$ the gradient vector is defined by,

Definition 159 *Let assume that the function f defined by $u = f(x, y, z)$ has partial derivatives f_x, f_y, f_z.*

$$\operatorname{grad} f = f_x i + f_y j + f_z k$$

Let we suppose

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

then we can write

$$\begin{aligned}\operatorname{grad} f &= \nabla f \\ &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\end{aligned}$$

Example 160 *Find the gradient vector field of the $f(x, y) = x^2 \sin(5y)$.*

Solution 161

$$\nabla f = \left\langle 2x \sin(5y), 5x^2 \cos(5y)\right\rangle$$

Definition 162 *A vector field $\overrightarrow{F}$ is called a conservative vector field if there exists a function f such that $\overrightarrow{F} = \nabla f$. If $\overrightarrow{F}$ is a conservative vector field then the function, f , is called a potential function for $\overrightarrow{F}$.*

Definition 163 *Given the vector field $\overrightarrow{F} = P\overrightarrow{i} + Q\overrightarrow{j} + R\overrightarrow{k}$ the curl is defined to be,*

$$\operatorname{curl} \overrightarrow{F} = (R_y - Q_z)\overrightarrow{i} i + (P_z - R_x)\overrightarrow{j} + (Q_x - P_y)\overrightarrow{k}.$$

$$\operatorname{curl}\overrightarrow{F} = \nabla \times \overrightarrow{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Definition 164 *Given the vector field $\overrightarrow{F} = P\overrightarrow{i} + Q\overrightarrow{j} + R\overrightarrow{k}$ the divergence is defined to be* $\operatorname{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

Example 165 *Compute* $\operatorname{div} F$ *for* $\overrightarrow{F} = x^2yi + xyzj - x^2y^2k$

Solution 166

$$\operatorname{div} F = \frac{\partial}{\partial x}\left(x^2y\right) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}\left(-x^2y^2\right) = 2xy + xz$$

Remark 167 *If the* $\operatorname{div} F = 0$ *for the vector field F then it is called that this vector field has free divergence.*

Let's recall the equation of a plane that contains the point (x_0, y_0, z_0) with normal vector $\overrightarrow{n} = (a, b, c)$ is given by,

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

When we introduced the gradient vector in the section on directional derivatives we gave the following fact.

Remark 168 *The gradient vector $\nabla f(x_0, y_0)$ is orthogonal (or perpendicular) to the level curve $f(x, y) = k$ at the point (x_0, y_0) . Likewise, the gradient vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface $f(x, y, z) = k$ at the point (x_0, y_0, z_0) .*

So, the tangent plane to the surface given by $f(x, y, z) = k$ at the point (x_0, y_0, z_0) has the equation,

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

The equation of the normal line is,

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t\,\nabla f\,(x_0, y_0, z_0)$$

Example 169 *Find the tangent plane and normal line to $x^2 + y^2 + z^2 = 30$ at the point $(1, -2, 5)$.*

Solution 170 $F\,(x, y, z) = x^2 + y^2 + z^2$

$$\nabla F = \langle 2x, 2y, 2z \rangle$$
$$\nabla F\,(1, -2, 5) = \langle 2, -4, 10 \rangle$$

The tangent plane is then,

$$2(x - 1) - 4(y + 2) + 10(z - 5) = 0$$

The normal line is,

$$\vec{r}(t) = \langle 1, -2, 5 \rangle + t\,\langle 2, -4, 10 \rangle = \langle 1 + 2t, -2 - 4t, 5 + 10t \rangle\,.$$

8.7 Change Of Variables

We call the equations that define the change of variables a transformation. Also, we will typically start out with a region, $\mathbb{R}$, in xy -coordinates and transform it into a region in uv -coordinates.

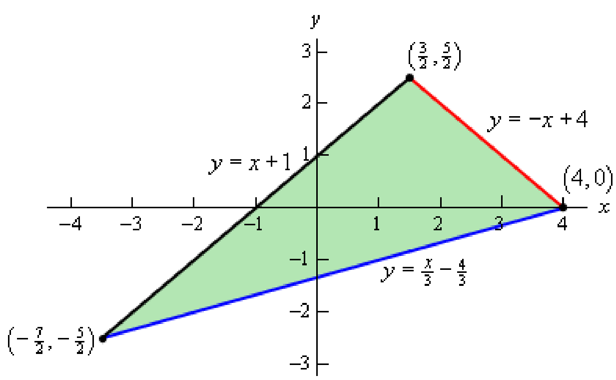
Example 171 *Determine the new region that we get by applying the given transformation to the region* $\mathbb{R}$.

a *R is the ellipse* $x^2 + \frac{y^2}{36} = 1$ *and the transformation is* $x = \frac{u}{2}, y = \frac{3}{v}$.

b *R is the region bounded by* $y = -x + 4$, $y = x + 1$, *and* $y = \frac{x}{3} - \frac{4}{3}$ *and the transformation is* $x = \frac{1}{2}(u + v)$, $y = \frac{1}{2}(u - v)$.

a Homework

b We'll need to plug the transformation into the equation, however, in this case we will need to do it three times, once for each equation. Before we do that let's sketch the graph of the region and see what we've got.



So, we have a triangle.

Now, let's go through the transformation. We will apply the transformation to each edge of the triangle and see where we get. Let's do $y = -x+4$ first. Plugging in the transformation gives,

$$\begin{aligned}\frac{1}{2}(u - v) &= -\frac{1}{2}(u + v) + 4 \\ u - v &= -u - v + 8 \\ 2u &= 8 \\ u &= 4\end{aligned}$$

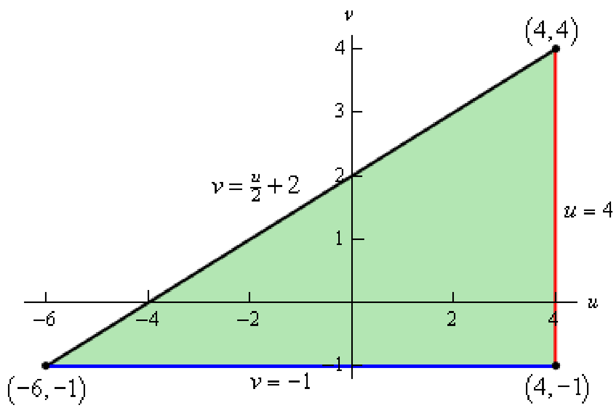
Now let's take a look at $y = x + 1$,

$$\begin{aligned}\frac{1}{2}(u - v) &= \frac{1}{2}(u + v) + 1 \\ u - v &= u + v + 2 \\ -2v &= 2 \\ v &= -1\end{aligned}$$

Finally, let's transform $y = \frac{x}{3} - \frac{4}{3}$,

$$\begin{aligned} \frac{1}{2}(u-v) &= \frac{1}{3}\left(\frac{1}{2}(u+v)\right) - \frac{4}{3} \\ 3u - 3v &= u + v - 8 \\ 4v &= 2u + 8 \\ v &= \frac{u}{2} + 2 \end{aligned}$$

Let's take a look at the new region that we get under the transformation.



Definition 172 *The Jacobian of the transformation* $x = g(u,v)$, $y = h(u,v)$ *is*

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

The Jacobian of the transformation $x = g(u,v,w)$, $y = h(u,v,w)$ *and* $z = k(u,v,w)$*is*

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$