8.8 DOUBLE INTEGRALS

Definition 173 *Let $f(x,y)$ be a continuous function defined on a bounded region B in the $xy-$plane and let*

$$P = \{B_i : 1 \leq i \leq n\}$$

be a partition of B by lines parallel to the coordinate axes, and define the norm of P as $\|P\| = \max\limits_{1\leq i\leq n} d_i$ where

$$d_i = \sup\{d(x,y) : x, y \in B_i\},$$

$$\|P\| = \max_{1\leq i\leq n}\{d(B_1), d(B_2), \cdots, d(B_n)\}.$$

For each $1 \leq i \leq n$ pick any point $(x_i, y_i) \in B_i$ and form the following sum

$$\sum_{i=1}^{n} f(x_i, y_i)\nabla A_i$$

where ∇A_i is the area of B_i. Such a sum is called an approximating sum or Riemann sum. Roughly speaking the double integral

$$\iint\limits_B f(x,y)\,dA$$

of f over B is defined to be the limit

$$\lim_{\|P\|\to 0}\sum_{i=1}^{n} f(x_i, y_i)\nabla A_i$$

In this case, we say that f is integrable on B.

Theorem 174 *Suppose that f is integrable over the rectangle $R = \{(x,y)|a \leq x \leq b$ and $c \leq y \leq d\}$. Then we can write the double integral of f over R as either of the iterated integrals:*

$$\iint\limits_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dydx = \int_c^d\int_a^b f(x,y)\,dxdy.$$

Example 175 *If $R = \{(x,y)|0 \leq x \leq 1$ and $0 \leq y \leq 2\}$,then evaluate $\iint\limits_R (x-y+1)\,dA$.*

Solution 176

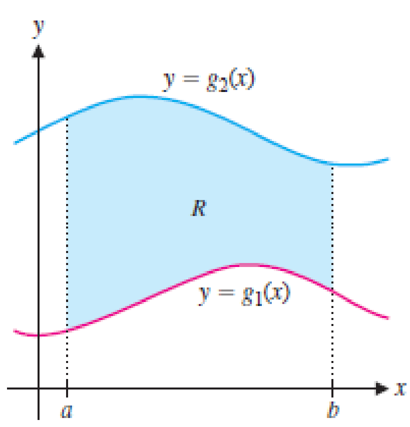
$$\begin{aligned}\iint_R (x-y+1)\,dA &= \int_0^1\int_0^2 (x-y+1)\,dydx \\ &= \int_0^1 \left(xy-\frac{y^2}{2}+y\right)\Big|_{y=0}^{y=2} dx \\ &= \int_0^1 2xdx = x^2|_{x=0}^{x=1} = 1\end{aligned}$$

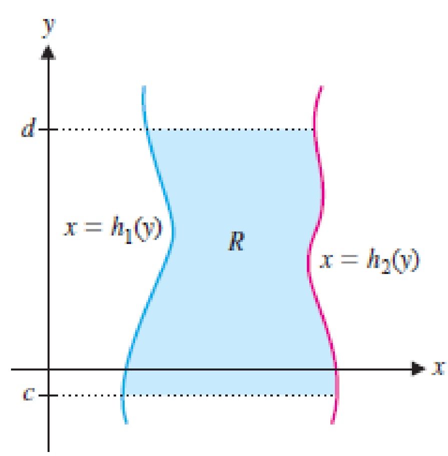
We leave it as an exercise to show that you get the same value by integrating first with respect to x, that is, that

$$\iint_R (x-y+1)\,dA = \int_0^2\int_0^1 (x-y+1)\,dxdy = 1$$

Theorem 177 *(Fubini's Theorem) Suppose that f is continuous on the region R defined by $R=\{(x,y)|a\le x\le b$ and $g_1(x)\le y\le g_2(x)\}$, for continuous functions g_1 and g_2 where $g_1(x)\le g_2(x)$, for all x in $[a,b]$. Then*

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dydx.$$





Theorem 178 *(Fubini's Theorem) Suppose that f is continuous on the region R defined by $R=\{(x,y)|c\le y\le d$ and $h_1(y)\le x\le h_2(y)\}$, for continuous*

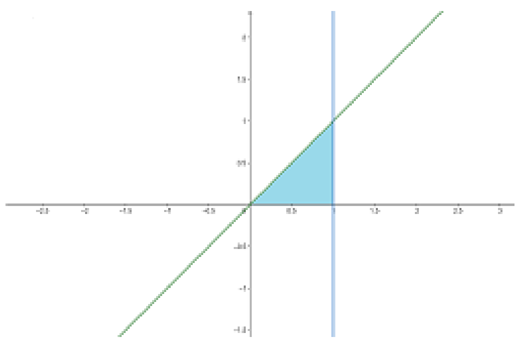
functions h_1 *and* h_2 *where* $h_1(y) \leq h_2(y)$, *for all* y *in* $[c, d]$. *Then*

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx dy.$$

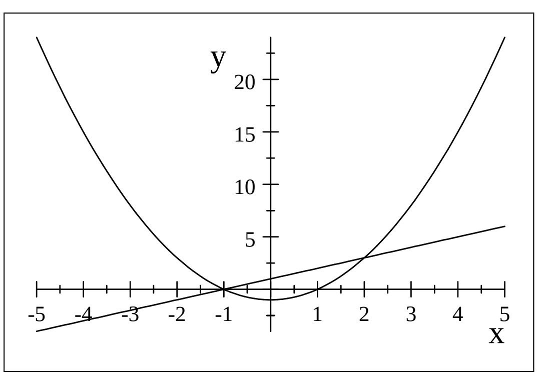
Example 179 *If* $R = \{(x, y) | 0 \leq x \leq 1$ *and* $0 \leq y \leq x\}$,*then evaluate* $\iint_R (x - y + 1)\, dA$.

Solution 180

$$\begin{aligned} \int_0^1 \int_0^x (x - y + 1)\, dy dx &= \int_0^1 \left(xy - \frac{y^2}{2} + y \right) |_{y=0}^{y=x} dx \\ &= \int_0^1 \left(\frac{x^2}{2} - x \right) dx = \left(\frac{x^3}{6} - \frac{x^2}{2} \right) |_{x=0}^{x=1} = \frac{-1}{3} \end{aligned}$$



Example 181 *Evaluate* $\iint_R (x + y)\, dA$ *where* R *be the region bounded by the line* $y = x + 1$ *and the curve* $y = x^2 - 1$.



Solution 182

$$\begin{aligned}
\iint\limits_R (x+y)\,dydx &= \int_{-1}^{2}\int_{y=x^2-1}^{y=x+1}(x+y)\,dydx = \int_{-1}^{2}\left(xy+\frac{y^2}{2}\right)_{y=x^2-1}^{y=x+1}dx\\
&= \int_{-1}^{2}\left(x\left(x+1\right)+\frac{\left(x+1\right)^2}{2}-x\left(x^2-1\right)-\frac{\left(x^2-1\right)^2}{2}\right)dx\\
&= \frac{99}{20}
\end{aligned}$$

8.8.1 Properties of Double Integrals

We list here three properties of double integrals

1. $\iint\limits_R kf(x,y)\,dA = k\iint\limits_R f(x,y)\,dA$ for any $k \in \mathbb{R}$.

2. $\iint\limits_R [f(x,y) \pm g(x,y)]\,dA = \iint\limits_R f(x,y)\,dA \pm \iint\limits_R g(x,y)\,dA$

3. If $f(x,y) \geq 0$ on R then $\iint\limits_R f(x,y)\,dA \geq 0$

4. If $f(x,y) \geq g(x,y)$ on R then $\iint\limits_R f(x,y)\,dA \geq \iint\limits_R g(x,y)\,dA$

Example 183 *Evaluate the iterated integral* $\int_0^1\int_y^1 e^{x^2}\,dxdy$.

Solution 184 *First, note that we cannot evaluate the integral the way it is presently written, as we don't know an antiderivative for* e^{x^2}. *If we switch the order of integration, the integral becomes quite simple.*

$$\begin{aligned}\int_0^1\int_y^1 e^{x^2}\,dxdy &= \int_0^1\int_0^x e^{x^2}\,dydx = \int_0^1 \left[e^{x^2}y\right]_{y=0}^{y=x} dx = \int_0^1 \left[e^{x^2}x\right] dx \\ &= \frac{1}{2}e^{x^2}|_{x=0}^{x=1} = \frac{1}{2}(e-1).\end{aligned}$$