Solution 223 *It is usefull to pass spherical coordinates.*

$$\begin{aligned}
&\int\limits_{-1}^{1}\int\limits_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int\limits_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}}\frac{1}{1+x^2+y^2+z^2}dzdydx\\
=&\int_0^{2\pi}\int_0^{\pi}\int_0^1\frac{1}{1+\rho^2}\rho^2\sin\phi d\rho d\phi d\theta\\
=&(4\pi-\pi^2)
\end{aligned}$$

Exercise 224 $\int_0^1\int_0^x\int_0^{\sqrt{x^2+y^2}}\frac{(x^2+y^2)^{3/2}}{x^2+y^2+z^2}dzdydx$. *(Answer* $\frac{\pi}{12}$*)*

Exercise 225 $\int_{-1}^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}}\sqrt{x^2+y^2+z^2}dzdydx$. *(Answer* $\pi\left(1-\frac{\sqrt{2}}{2}\right)$*)*

8.10.5 Applications of Triple Integrals

Volume We can use cyldrical and spherical coordinates to find the volume of the solid.

$$V=\iiint\limits_Q dV=\iiint\limits_Q rdzdrd\theta$$

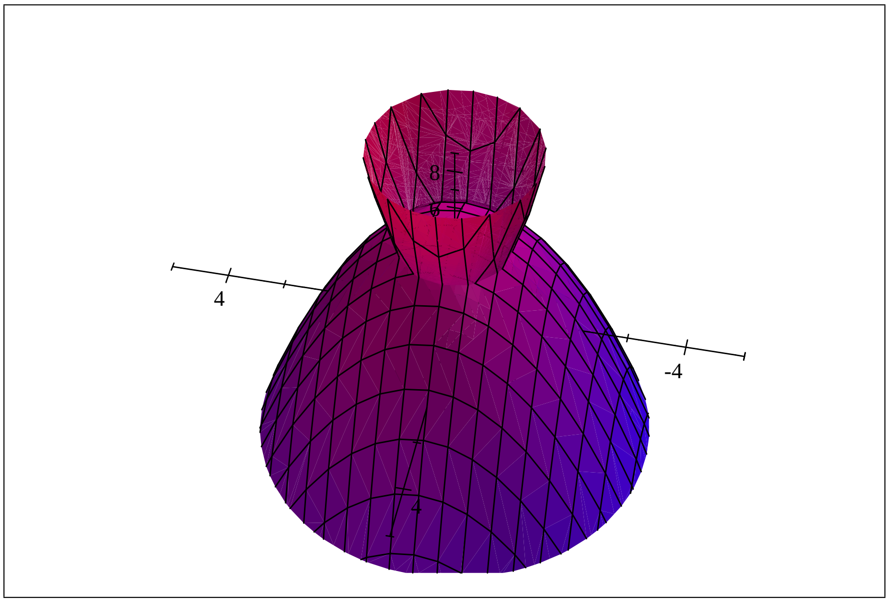
$$V=\iiint\limits_Q dV=\iiint\limits_Q \rho^2\sin\phi d\rho d\theta d\phi$$

Example 226 *Find the volume of the sphere of radius a, by using triple integral.*

Solution 227 *To evaluate the volume we will use spherical coordinates.*

$$\begin{aligned}
V&=\iiint\limits_{x^2+y^2+z^2\le a^2}dV=\int_0^{2\pi}\int_0^{\pi}\int_0^a\rho^2\sin\phi d\rho d\theta d\phi\\
&=\int_0^{2\pi}\int_0^{\pi}\frac{\rho^3}{3}\sin\phi\ |_0^a\ d\theta d\phi=\int_0^{2\pi}\int_0^{\pi}\frac{a^3}{3}\sin\phi d\theta d\phi\\
&=\frac{a^3}{3}\int_0^{2\pi}-\cos\phi\ |_0^{\pi}\ d\phi=\frac{2a^3}{3}\phi\ |_0^{2\pi}=\frac{4a^3\pi}{3}\ units^3
\end{aligned}$$

Example 228 *Find the volume of solid bounded by the paraboloids* $z=5-x^2-y^2$ *and* $z=4x^2+4y^2$.

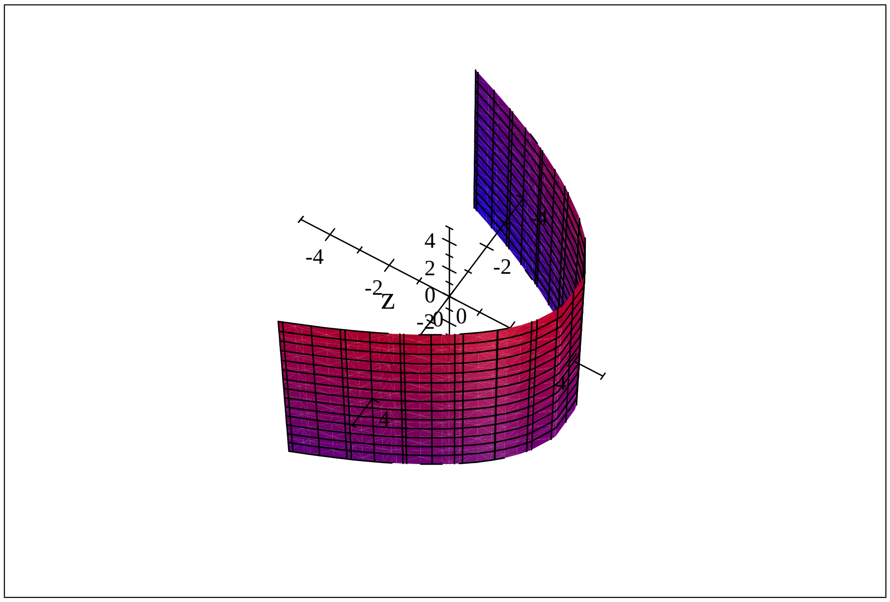


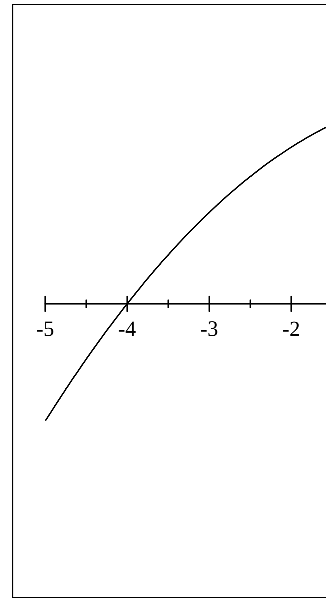
Solution 229

$$\begin{aligned} 5-x^2-y^2 &= 4x^2+4y^2 \\ x^2+y^2 &= 1 \end{aligned}$$

$$\begin{aligned} V &= \int\int_{x^2+y^2\leq 1} \int_{z=4x^2+4y^2}^{z=5-x^2-y^2} dzdydx = \int\int_{x^2+y^2\leq 1} \left[5-5x^2-5y^2\right] dxdy \\ &= 5\int_0^{2\pi}\int_0^1 \left(1-r^2\right) rdrd\theta = \frac{5}{2}\pi \ units^3 \end{aligned}$$

Example 230 *Find the volume of solid bounded by the cylinder* $4y=16-x^2$ *and the planes* $y=0,\ z=0,\ z=3.$

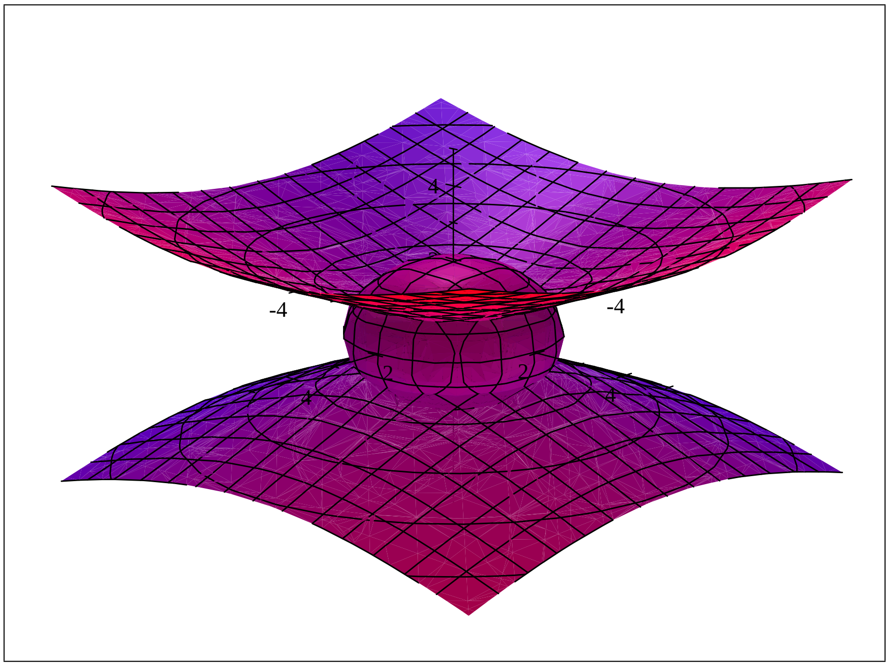




Solution 231

$$\begin{aligned} V &= \iint_R \int_{z=0}^{z=3} dz dy dx = 3 \int_{x=-4}^{x=4} \int_{y=0}^{y=\frac{16-x^2}{4}} dy dx \\ &= 3 \int_{-4}^{4} \frac{16-x^2}{4} dx = \frac{3}{4} \left(16x - \frac{x^3}{3}\right)_{-4}^{4} \\ &= 64 \ un^3 \end{aligned}$$

Example 232 *Find the volume of solid that lies inside the sphere* $x^2+y^2+z^2 = 4$ *and outside the cone* $x^2+y^2 = 3z^2$



Solution 233 *We will use spherical coordinates.*

$$\begin{aligned} 3\rho^2\cos^2\theta &= \rho^2\sin^2\theta\cos^2\phi + \rho^2\sin^2\theta\sin^2\phi \\ &= \rho^2\sin^2\theta \\ 3 &= \frac{\sin^2\theta}{\cos^2\theta} \Longrightarrow \tan\theta = \pm\sqrt{3} \end{aligned}$$

$$\begin{aligned} V &= \int_0^{2\pi}\int_{\pi/3}^{2\pi/3}\int_0^a \rho^2\sin\theta d\rho d\theta d\phi = \int_0^{2\pi}\int_{\pi/3}^{2\pi/3} \frac{a^3}{3}\sin\theta d\theta d\phi \\ &= \frac{a^3}{3}\int_0^{2\pi} (-\cos\theta)_{\pi/3}^{2\pi/3} d\phi = \frac{a^3}{3}\phi|_0^{2\pi} = \frac{2\pi a^3}{3} \ \ units^3 \end{aligned}$$

The Mass As we know the mass of an object in space is equal to the multiplication of the density and volume of this object.

$$M = \iiint_Q \sigma(x,y,z)\,dxdydz$$

where $\sigma(x,y,z)$ is density at the point (x,y,z).

Example 234 *Suppose that an object is placed between the upper hemisphere $x^2+y^2+z^2 = a^2$ and the plane $z = 0$. If the density of this object at its each point changes propartionally with the distance of this point from the origin. Evaluate the mass os this object.*

Solution 235

$$\sigma(x,y,z) = \alpha\sqrt{x^2+y^2+z^2}$$

$$\begin{aligned} M &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} \alpha\rho\rho^2 \sin\theta d\rho d\theta d\phi = \alpha\int_0^{2\pi}\int_0^{\pi/2} \frac{a^4}{4}\sin\theta d\theta d\phi \\ &= \alpha\frac{a^4}{4}\int_0^{2\pi} (-\cos\theta)_0^{\pi/2} = \alpha\frac{a^4}{4}\int_0^{2\pi} d\phi = \alpha\frac{a^4}{2}\pi \end{aligned}$$

Center of The Mass Similar to double integrals it can be easily find that the coordinates of the center of mass.

$$\begin{aligned} \bar{x} &= \frac{1}{M}\iiint_Q x\sigma(x,y,z)\,dxdydz \\ \bar{y} &= \frac{1}{M}\iiint_Q y\sigma(x,y,z)\,dxdydz \\ \bar{z} &= \frac{1}{M}\iiint_Q z\sigma(x,y,z)\,dxdydz \end{aligned}$$

Example 236 *Let suppose the solid Q bounded below by $xy-$plane above by paraboloid $x^2+y^2 = z$ and on the side by the cylinder $x^2+y^2 = 4$. If an homegeneous object with constant density isplaced in Q then find the its coordinates of the center of mass.*

Solution 237 *Since the object is homogeneous and since it is symetric with respect to z, its coordinates of center of mass on z. So $\bar{x} = \bar{y} = 0$. $\sigma(x,y,z) = c$*

$$\begin{aligned} M &= \int\int_{x^2+y^2\leq 4}\int_0^{x^2+y^2} c\,dzdxdy = \int_0^{2\pi}\int_0^{2}\int_0^{x^2+y^2} crdzdrd\theta \\ &= 8\pi c \end{aligned}$$

$$\begin{aligned} \bar{z} &= \frac{1}{8\pi c}\int\int_{x^2+y^2\leq 4}\int_0^{x^2+y^2} zcdxdydz = \frac{1}{8\pi c}\int_0^{2\pi}\int_0^{2}\int_0^{x^2+y^2} zcrdzdrd\theta \\ &= \frac{4}{3} \end{aligned}$$

$$G = \left(0, 0, \frac{4}{3}\right)$$