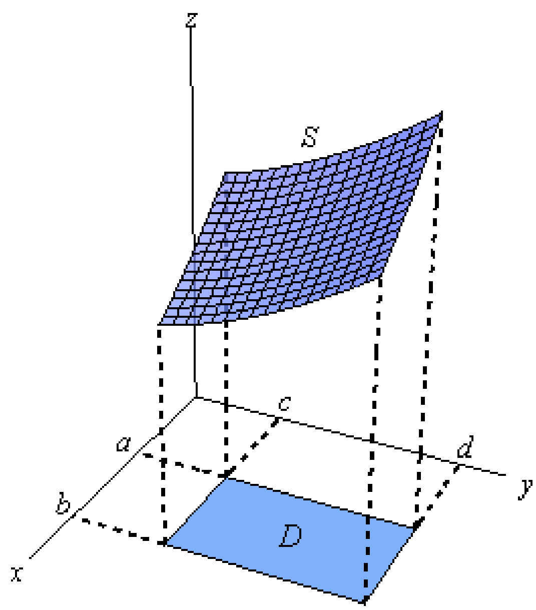
10 SURFACE INTEGRALS

Surface integrals are integrals that their integration parts are surfaces.In this section we will investigate first and second type of surface integrals and we will give applications of them.

10.1 First Type Surface Integrals

Let D is a region on $xy-$ plane and $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, $z = f(x, y)$ is a function having continuous partial derivatives. Now we suppose that S is the graph of this function.



$P_1 = \{S_1, S_2, \cdots, S_n\}$ is a partition of S. ΔS_k is the area of S_k $(k = 1, 2, \cdots, n)$ and assume that $(x_k, y_k, z_k) \in S_k$. If the limit

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} g(x_k, y_k, z_k) \Delta S_k$$

exists for the function

$$g : \underset{(x,y,z)\to}{S} \to \underset{g(x,y,z)}{\mathbb{R}}$$

then this limit called the surface integral of the function g on S and shown by

$$\iint_S g(x, y, z)\, ds.$$

$ds = \sqrt{1+f_x^2+f_y^2}dxdy$ and region D is the orthogonal projection of surface S on $xy-$plane.

$$I = \iint\limits_S g(x,y,z)\,ds = \iint\limits_D g(x,y,f(x,y))\sqrt{1+f_x^2+f_y^2}dxdy.$$

Example 286 *Evaluate* $\iint\limits_S (x+y+z)\,ds$ *where* S *is the partition of the plane* $2x+2y+z=2$ *in the first region.*

Solution 287 $z = f(x,y) = 2-2x-2y,\ z_x = -2,\ z_y = -2$

$$\begin{aligned}\iint\limits_S (x+y+z)\,ds &= \iint\limits_D (x+y+2-2x-2y)\sqrt{1+(-2)^2+(2)^2}dxdy \\ &= \iint\limits_D 3(2-x-y)\,dxdy = 3\int_0^1\int_0^{1-y} 3(2-x-y)\,dxdy \\ &= 2\end{aligned}$$

Remark 288 *If the equation of* S *is given by* $x = f(y,z)$ *and if the orthogonal projection of* S *on* yz*- plane is* G *then*

$$I = \iint\limits_S g(x,y,z)\,ds = \iint\limits_G g(f(y,z),y,z)\sqrt{1+f_y^2+f_z^2}dydz.$$

10.2 Integrals on Directed Surfaces (Surface Integrals of Vector Fields)

Let assume that the surface S has a tangent plane on every $(a,b,c)\in S$ except its boundary points. We can draw two normal vectors at each point $(a,b,c)\in S$, which are opposite directions. Consider the surface S with $z = f(x,y)$. $(g(x,y,z) := f(x,y)-z)$ The normal vectors of the surface $z = f(x,y)$ on the point (a,b,c) are given by:

$$\begin{aligned} N &= f_x i + f_y j - k \\ -N &= M = -f_x i - f_y j + k \end{aligned}$$

If we unitize these normals, we get

$$n = \frac{f_x i + f_y j - k}{\sqrt{1+f_x^2+f_y^2}} \text{ and } -n = m = \frac{-f_x i - f_y j + k}{\sqrt{1+f_x^2+f_y^2}}$$

If the unit normal vector directed to the upper i.e., ifits $k-$component is positive, we will use the m form as an unit vector and if te normal vector directed to below direction we will use the n as an unit normal vector. If a surface and its normal vector are given we called this surface as adirected surface.

Let suppose that $F = Pi + Qj + Rk$ is a vector field defined on a surface S. Then the integral

$$\iint_S F \cdot nds$$

is called a second type surface integral of F over the surface S. Sometimes it is called a Flux integral of F over the surface S. If the orthogonal projection of S on xy-pllane is B and the normal vector of S given with $n = \frac{f_x i + f_y j - k}{\sqrt{1+f_x^2+f_y^2}}$ then

$$\begin{aligned} \iint_S F \cdot nds &= \iint_B (Pi + Qj + Rk) \left(\frac{f_x i + f_y j - k}{\sqrt{1 + f_x^2 + f_y^2}} \right) \sqrt{1 + f_x^2 + f_y^2} dxdy \\ &= \iint_B (Pf_x + Qf_y - R)\, dxdy \end{aligned}$$

If the orthogonal projection of S on xy-pllane is B and the normal vector of S given with $-n = m = \frac{-f_x i - f_y j + k}{\sqrt{1+f_x^2+f_y^2}}$ then

$$\begin{aligned} \iint_S F \cdot nds &= \iint_B (-Pf_x - Qf_y + R)\, dxdy \\ &= -\iint_B (Pf_x + Qf_y - R)\, dxdy \end{aligned}$$

Example 289 *Evaluate integral of vector field $F = x^2 i + y^2 j + z^2 k$ over the surfaces, where S is a partition of cone $z = \sqrt{x^2 + y^2}$, $1 \leq z \leq 2$ and its normal is directed to outside of the cone. ($\iint_S F \cdot nds = ?$)*

Solution 290 *Since n is directed to the outside of cone, we use the formula $n = \frac{f_x i + f_y j - k}{\sqrt{1+f_x^2+f_y^2}}$.*

$$\begin{aligned} z &= f(x, y) = \sqrt{x^2 + y^2} \\ f_x &= \frac{x}{\sqrt{x^2 + y^2}} \text{ and } f_y = \frac{y}{\sqrt{x^2 + y^2}} \\ n &= \frac{\frac{x}{\sqrt{x^2+y^2}} i + \frac{y}{\sqrt{x^2+y^2}} j - k}{\sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}} = \frac{\frac{x}{\sqrt{x^2+y^2}} i + \frac{y}{\sqrt{x^2+y^2}} j - k}{\sqrt{2}} \end{aligned}$$

$$
\begin{aligned}
F \cdot n &= \frac{1}{\sqrt{2}} \left[\frac{x}{\sqrt{x^2+y^2}} i + \frac{y}{\sqrt{x^2+y^2}} j - k \right] \left(x^2 i + y^2 j + z^2 k \right) \\
&= \frac{1}{\sqrt{2}} \left(\frac{x^3}{\sqrt{x^2+y^2}} + \frac{y^3}{\sqrt{x^2+y^2}} - z^2 \right)
\end{aligned}
$$

$$
\begin{aligned}
\iint\limits_S F \cdot nds &= \iint\limits_B \frac{1}{\sqrt{2}} \left(\frac{x^3}{\sqrt{x^2+y^2}} + \frac{y^3}{\sqrt{x^2+y^2}} - z^2 \right) dxdy \\
&= \iint\limits_B \frac{1}{\sqrt{2}} \left(\frac{x^3}{\sqrt{x^2+y^2}} + \frac{y^3}{\sqrt{x^2+y^2}} - \left(x^2 + y \right) \right) dxdy \\
&= \int\limits_0^{2\pi} \int\limits_0^2 \left(\frac{r^3 \cos^3 \theta + r^3 \sin^3 \theta}{r} - r^2 \right) rdrd\theta \\
&= \int\limits_0^{2\pi} \int\limits_0^2 r^3 \left(\cos^3 \theta + \sin^3 \theta - 1 \right) drd\theta \\
&= \frac{15}{4} \int\limits_0^{2\pi} \left(\cos^2 \theta \cos \theta + \sin^2 \theta \sin \theta - 1 \right) d\theta \\
&= \frac{15}{4} \int\limits_0^{2\pi} \left[\left(1 - \sin^2 \theta \right) \cos \theta + \left(1 - \cos^2 \theta \right) \sin \theta - 1 \right] d\theta \\
&= \frac{15}{4} \left(\sin \theta - \frac{\sin^3 \theta}{3} - \cos \theta + \frac{\cos^3 \theta}{3} - \theta \right) |_0^{2\pi} \\
&= \frac{-15\pi}{2}
\end{aligned}
$$

Remark 291 *Let assume that the surface S is bounded by a curve C and te normal of S is n. By using the direction of S the curve C can be directed. First closed your right hand, the thumb of your hand is applied to the surface that its direction must be in the same direction of the surface normal. Other fingers direcrtion is the direction of the curve C. This direction of the curve C is called the reduced direction from the surface S. If the directed surface S is a combination of the surfaces $S_1, S_2, \cdots, S_m$ and the common curves of S_k $(k = 1, 2, \cdots, m)$ have opposite directions, then*

$$
\iint\limits_S F \cdot nds = \iint\limits_{S_1} F \cdot nds + \iint\limits_{S_2} F \cdot nds + \cdots + \iint\limits_{S_m} F \cdot nds.
$$

10.3 Fundamental Theorems of Surface Integrals

In this section we will give Stoke's Theorem and Divergence Theorem of surface integrals. Actually these theorems are generalizations of Greenn Theorem to the three dimensional spaces.

Theorem 292 (Stoke's Theorem) *Let S be an directed surface with boundary C which is piecewise smooth, and let*

$$F(x,y,z) = P(x,y,z)\,i + Q(x,y,z)\,j + R(x,y,z)\,k$$

be a vector field defined on a region containing S. If P, Q, R have continuous partial derivatives, then

$$\oint_C F\cdot dr = \iint_S (rotF)\cdot nds$$

where n is the normal of S.

Notation 293 *Let a vector field $F(x,y,z) = P(x,y,z)\,i + Q(x,y,z)\,j + R(x,y,z)\,k$ be such that P, Q, R have continuous partial derivatives in same region. The* curl (rot) *of F is given by*

$$\begin{aligned} \nabla\times F &= \operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= (R_y - Q_z)\,i + (P_z - R_x)\,j + (Q_x - P_y)\,k \end{aligned}$$

Example 294 *Assume that $F(x,y,z) = 4yi + xj + 2zk$, S be the above part of the sphere $x^2+y^2+z^2=a^2$. The normal vector n is directed tot he outside of the sphere. Evaluate $\iint_S (\operatorname{curl} F)\cdot nds$.*

Solution 295 $z = f(x,y) = \sqrt{a^2-x^2-y^2}$

$$\begin{aligned} C &: r(t) = a\cos t i + a\sin t j,\ 0\le t\le 2\pi \\ r'(t) &= -a\sin t i + a\cos t j, \end{aligned}$$

$$\begin{aligned}
\iint_S (\operatorname{curl} F)\cdot n ds &= \oint_C F\cdot dr \\
&= \int_0^{2\pi} (4a\sin ti + a\cos tj + 0k)(-a\sin ti + a\cos tj)\,dt \\
&= \int_0^{2\pi} \left(-4a^2\sin^2 t + a^2\cos^2 t\right) dt = \int_0^{2\pi} \left(-4a^2\sin^2 t + a^2\left(1-\sin^2 t\right)\right) dt \\
&= \int_0^{2\pi} \left(-5a^2\sin^2 t + a^2\right) \\
&= \frac{-5}{2}a^2 2\pi + a^2 2\pi = -3a^2\pi
\end{aligned}$$

Example 296 *Let* $F(x,y,z) = e^x\sin yi + (e^x\cos y - z)j + yk$*,* S *be the part of the cone* $z = 1-\sqrt{x^2+y^2}$ *which remains on the top of the* $xy-$*plane and* n *normal vector is oriented to the positive direction. Then evaluate the integral of* F *over the boundary curve of* S*.* $\left(\int_{\partial S} F\cdot dr = ?\right)$

Solution 297 *It is diffucult to evaluate this integral by classical way like the previous example. So we wil use Stokes Theorem.*

$$\begin{aligned}
\operatorname{curl} F &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&= (R_y - Q_z)i + (P_z - R_x)j + (Q_x - P_y)k \\
&= (1+1)i + (0-0)j + (e^x\cos y - e^x\cos y) = 2i
\end{aligned}$$

$$n = \frac{-f_x i - f_y j + k}{\sqrt{1+f_x^2+f_y^2}} = \frac{x}{\sqrt{2}\left(x^2+y^2\right)^{1/2}}i + \frac{x}{\sqrt{2}\left(x^2+y^2\right)^{1/2}}j + \frac{1}{\sqrt{2}}k$$

$$\begin{aligned}
\int_{\partial S} F\cdot dr &= \iint_S \operatorname{curl} F\cdot n ds \\
&= \iint_S 2i\left(\frac{x}{\sqrt{2}\left(x^2+y^2\right)^{1/2}}i + \frac{x}{\sqrt{2}\left(x^2+y^2\right)^{1/2}}j + \frac{1}{\sqrt{2}}k\right) ds \\
&= \iint_B \frac{2x}{\sqrt{2}\left(x^2+y^2\right)}\sqrt{2}dxdy \\
&= \int_0^{2\pi}\int_0^1 \frac{2r\cos\theta}{r} r dr d\theta = \int_0^{2\pi} \cos\theta d\theta = \sin\theta\big|_0^{2\pi} = 0
\end{aligned}$$

Definition 298 *Let D is a region in $xyz-$space. If the region D is between the graphs of functions f_1, f_2 defined in a region B_1 on xy-plane, between the graphs of functions $g_{1,}$ g_2 defined in a region in B_2 on xz-plane and between the graphs of functions h_1, h_2 defined in a region B_3 on yz-plane. Then D is called a simple solid region.*

Let D be a simple solid region, S is the boundary surface of D with positive orientation and n is the normal of this space with the direction outward from D. Let

$$F(x, y, z) = P(x, y, z)\, i + Q(x, y, z)\, j + R(x, y, z)\, k$$

be a vector field whose components have continuous first order partial derivatives on D. Then

$$\iint\limits_S F \cdot n ds = \iiint\limits_D \operatorname{div} F dv$$

where $\operatorname{div} F = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

Example 299 *Find the integral of vector field $F = xi + 2yj + 3zk$ over the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$*

Solution 300

$\operatorname{div} F = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 1 + 2 + 3 = 6$

$$\begin{aligned}
\iint\limits_S F \cdot n ds &= \iiint\limits_D \operatorname{div} F dv \\
&= 6 \iiint\limits_D dx dy dz \\
&= 6 \frac{4}{3} \pi abc
\end{aligned}$$

10.4 Applications of Surface Integrals

10.4.1 Calculation of Surface Area

The area of surface S is given by

$$A = \iint\limits_S ds$$

where $ds = \sqrt{1 + (f_x)^2 + (f_y)^2} dx dy$ and $z = f(x, y)$

Exercise 301 *Calculate the surface area of the sphere with radius a.*