3. Heat Equation

In this section we will deal with the partial differential equation

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0$$

which is a classic example of parabolic type equations with independent variables x and t. This equation, known as the one dimensional heat equation, appears during the study of heat conduction in objects. k is a constant that depends on the degree of conductivity of the object under consideration and is called *the diffusion constant*.

3.1. Heat Conduction Problem

Consider a homogeneous straight bar of length L. Let's assume that this bar, which is located along the $0 \leq x \leq L$ range on the x-axis, is thin enough and this situation ensures that the heat distribution over the vertical section of the bar, corresponding to any moment t, can be taken equally.

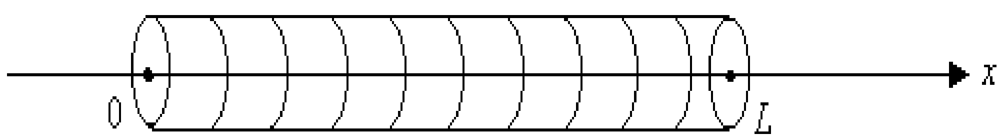

Figure 3.1. Heat conduction in a thin bar

Also, let's assume that the lateral surface of this bar is insulated so that there is no heat loss across the surface. In this case, the heat flow through the bar will only be in the x-axis direction. Let us denote by $u(x,t)$ the heat of the vertical section of the bar at point x at any time t. In this case, the function $u(x,t)$, which gives the heat distribution in the bar, will be the solution of the initial and boundary value problem given below.

3.2. Initial and Boundary Value Problem

Consider the one-dimensional heat equation

$$u_t - ku_{xx} = 0 \quad ; \qquad 0 < x < L \quad , \qquad t > 0 \tag{1}$$

$$u(x,0) = f(x) \quad ; \qquad 0 \leq x \leq L \tag{2}$$

with initial condition and

$$u(0,t) = 0 \quad , \qquad u(L,t) = 0 \quad ; \qquad t \geq 0 \tag{3}$$

boundary condition. While solving such a problem, we will also explain a very useful and powerful method that can be used to solve initial value or initial and boundary value or boundary value problems given for a wide class of partial differential equations of the hyperbolic, parabolic or elliptic type. In order to create the basic steps in this method known as the method of separation of variables, we will solve the initial and boundary value problems of the heat equations described with (1), (2) and (3) above with this method. We seek a solution of equation (1) in the form of

$$u(x,t) = X(x)T(t). \tag{4}$$

In the solution (4), X is only a function of x, and T is only a function of t. If partial derivatives u_{xx} and u_t are taken from (4) and they are written in (1), we have

$$X(x)T^{'}(t) - kX^{''}(x)T(t) = 0.$$

$$T^{'}(t) = \frac{dT}{dt} \quad , \quad X^{''}(x) = \frac{d^2X}{dx^2}$$

If we divide last expression by kTX and write it by separating its variables, we obtain

$$\frac{T^{'}}{kT} = \frac{X^{''}}{X} \tag{5}$$

The left side of equation (5) depends only on the variable t, the right side depends only on the variable x. If we take the partial derivatives of both sides of (5) with respect to x and t respectively, we see that the derivative of the first side with respect to x and the derivative of the second side with respect to t are zero. This is only possible if both sides of (5) are equal to a constant. If we denote the constant with $-\lambda$, we have

$$\frac{T^{'}}{kT} = \frac{X^{''}}{X} = -\lambda, \tag{6}$$

which gives ordinary differential equations as follows

$$X^{''} + \lambda X = 0 \tag{7}$$

and

$$T^{'} + \lambda kT = 0. \tag{8}$$

Thus, the partial differential equation (1) is replaced by the ordinary differential equations (7) and (8), which contain an parameter λ. From (3), we have

$$\begin{aligned} u(0,t) &= X(0)T(t) = 0 \ \Rightarrow \ X(0) = 0 \\ u(L,t) &= X(L)T(t) = 0 \ \Rightarrow \ X(L) = 0. \end{aligned}$$

It then turns out that the function X must be the solution to the Sturm Liouville problem on $0 \leq x \leq L$

$$\left. \begin{array}{lcl} X^{''} + \lambda X = 0 & ; & 0 < x < L \\ X(0) = 0 & , & X(L) = 0 \end{array} \right\}. \tag{9}$$

The eigenvalues are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \quad ; \qquad n = 1, 2, ... \tag{10}$$

and the corresponding eigenfunctions are

$$X_n(x) = \sin\frac{n\pi x}{L} \quad ; \qquad n = 1, 2, ... \tag{11}$$

On the other hand, for

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad (n \geq 1),$$

a solution of equation (8) is obtained as

$$T_n(t) = e^{-k\lambda_n t}. \tag{12}$$

Thus,

$$u_n(x,t) = e^{-k\lambda_n t}\sin\frac{n\pi x}{L} \quad ; \qquad n = 1, 2, ... \tag{13}$$

In order to obtain a solution to the problem given by (1), (2) and (3), let's consider a series of functions (13) in the form of

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-k\lambda_n t}\sin\frac{n\pi x}{L} \tag{14}$$

and let us determine the coefficients b_n to satisfy the initial conditions (2). If we take $t = 0$ in (14) and keep (2) in mind, we see that the coefficients b_n must satisfy the relation

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad ; \qquad 0 \leq x \leq L. \tag{15}$$

Since this is Fourier sine series of $f(x)$ in the half interval $[0, L]$, the Fourier coefficients b_n are obtained by the formula

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \quad ; \qquad n = 1, 2, ...$$

The $u(x,t)$ function defined by the series which is found by replacing the b_n in (14), becomes the desired solution of our problem.

Example 1. Find the solution of the initial and boundary value problem given below

$$\begin{aligned} u_t - u_{xx} &= 0 \quad ; \qquad 0 < x < \pi \quad , \qquad t > 0 \\ u(x,0) &= \sin x \quad ; \qquad 0 \leq x \leq \pi \\ u(0,t) &= u(\pi,t) = 0 \quad ; \qquad t \geq 0 \end{aligned}$$

Solution: When the method of separation of variables is applied to the given problem, it will be seen that it has a solution in the form of

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-k\lambda_n t} \sin \frac{n\pi x}{L}\,.$$

If we take

$$k = 1 \quad , \quad L = \pi \quad , \quad \lambda_n = \frac{n^2\pi^2}{\pi^2} = n^2$$

since the coefficients b_n are found as follows

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin x \sin nx dx = \begin{cases} 1 & , \quad n = 1 \\ 0 & , \quad n > 1 \end{cases} ,$$

the desired solution to the problem is obtained as

$$u(x,t) = b_1 e^{-\lambda_1 t} \sin x + \sum_{n>1}^{\infty} b_n e^{-k\lambda_n t} \sin \frac{n\pi x}{L} = e^{-t} \sin x\,.$$

Example 2. Find the solution of the problem

$$\begin{array}{lllll} u_t - u_{xx} = 0 & ; & 0 < x < \pi & , & t > 0 \\ u(x,0) = x(\pi - x) & ; & 0 \leq x \leq \pi & & \\ u(0,t) = 0 & , & u(\pi,t) = 0 & ; & t \geq 0 \end{array} .$$

Solution: The given problem is the initial and boundary value problem with $f(x) = x(\pi - x)$ in the interval $[0,\pi]$ for the heat equation with $k = 1$. When the separation of variables method is applied, it will be seen that it has a solution as

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx\,.$$

If the coefficients b_n are calculated, they are found as

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx dx = 2\int_0^{\pi} x \sin nx dx - \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx dx \\ &= 4\frac{1-(-1)^n}{\pi n^3} = \begin{cases} 0 & , \quad for\ n = 2k \\ \frac{8}{\pi(2k-1)^3} & , \quad for\ n = 2k-1 \end{cases} \end{aligned}$$

Thus, the desired solution is obtained

$$u(x,t) = \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{e^{-(2n-1)^2 t}}{(2n-1)^3} \sin(2n-1)x$$