4. Laplace Equation

In this section, Laplace equation with two independent variables

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

will be discussed and boundary value problems related to this equation will be examined. The equation (1) is usually expressed in the form

$$\Delta u = 0$$

with the help of Laplace operator which is given by

$$\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Some of the partial differential equations which are expressed with the help of the Laplace operator and also which are of great importance in Mathematical Physics are as follows.

A. Two-dimensional Laplace equation

$$\Delta u = 0$$

B. Poisson equation

$$\Delta u = q(x, y)$$

C. Helmoltz equation

$$\Delta u + \lambda u = 0 \ , \qquad \lambda \ \text{ is a positive constant}$$

D. Schrödinger equation (time independent)

$$\Delta u + [\lambda - q(x, y)]\, u = 0$$

E. Two-dimensional heat equation

$$\Delta u = \frac{1}{k}\frac{\partial u}{\partial t}$$

4.1. Boundary Value Problem

A boundary value problem is a problem of finding a given partial differential equation with certain boundary conditions. They are physically time-independent problems that only involve space coordinates. There are three types of boundary value problems and they are defined as follows.

I. Dirichlet Problem

It is the problem of finding a function $u(x, y)$ that is harmonic in a region D and satisfies the boundary condition

$$u = f(x, y) \tag{2}$$

on the boundary C of D where f is a known function which is defined on the boundary C.

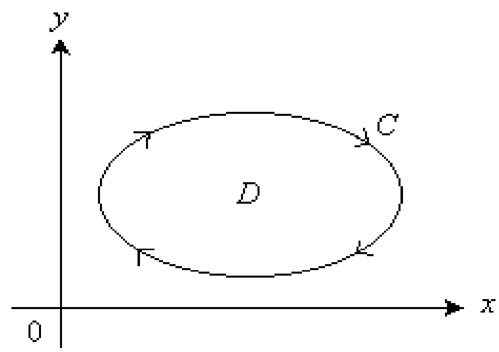

Figure 4.1. Planar region and boundary curve

II. Neumann Problems

It is the problem of finding a function $u(x, y)$ that is harmonic in a region D and satisfies the boundary condition

$$\frac{\partial u}{\partial n} = f(x, y) \tag{3}$$

on the boundary C of D. Here, $\dfrac{\partial u}{\partial n}$ defines the outer normal derivative of u on C.

III. Robin Problem (Mixed Boundary Value Problem)

It is the problem of finding a function $u(x, y)$ that is harmonic in a D region and satisfies the boundary condition

$$\frac{\partial u}{\partial n} + h(x, y)u = g(x, y) \tag{4}$$

on the boundary C of D. Here h and g are known functions given before.

4.2. Dirichlet Problem for a Rectangle

In this section, we will obtain the solution of the Dirichlet problem in a rectangular region R, which is a simple region of the plane. The best way to do this is to use the method of separation of variables. Let R be an open rectangular region in the xy-plane

$$R = \{(x, y) : \quad 0 < x < a \quad , \quad 0 < y < b\} .$$

Our problem is to find a function $u(x, y)$ that satisfies the partial differential equation

$$u_{xx} + u_{yy} = 0 \quad , \qquad \text{Inside } R \tag{5}$$

and the boundary conditions (Figure 7.2)

$$\left.\begin{array}{lllll} u(0,y) = 0 & , & u(a,y) = 0 & ; & 0 \le y \le b \\ u(x,0) = 0 & , & u(x,b) = f(x) & ; & 0 \le x \le a \end{array}\right\} . \tag{6}$$

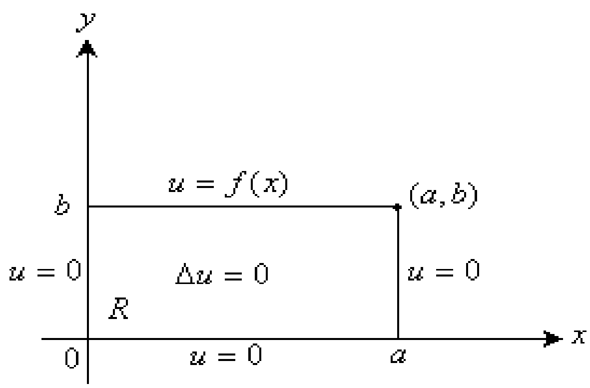

Figure 4.2. Dirichlet problem for a rectangle

According to the method of separation of variables, a trivial solution in the form of $u(x, y)$ is obtained as follows

$$u(x, y) = X(x)Y(y) \tag{7}$$

This solution must satisfy equation (5) and boundary conditions (6). We have the following solution

$$u(x, y) = \sum_{n=1}^{\infty} b_n \, \frac{\sinh \frac{n\pi y}{a}}{\sinh \frac{n\pi b}{a}} \, \sin \frac{n\pi x}{a} . \tag{8}$$

The values of the coefficients b_n in (8) are calculated by

$$b_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} \, dx \quad ; \qquad n = 1, 2, ... \tag{9}$$

Remark: The general Dirichlet problem defined in a rectangular region $R = \{(x, y) : 0 < x < a, \ 0 < y < b\}$ is as follows.

$$\left.\begin{array}{lllll} u_{xx} + u_{yy} = 0 & , & \text{Inside } R & & \\ u(x,0) = f_1(x) & , & u(x,b) = f_2(x) & ; & 0 \le x \le a \\ u(0,y) = f_3(y) & , & u(a,y) = f_4(y) & ; & 0 \le y \le b \end{array}\right\} \tag{10}$$

For $i = 1, 2, 3, 4$, when the others f_i are zero except for one of f_i, the solution u_i $(1 \leq i \leq 4)$ of the problem in (10) is found by using the above method, and then the solution of (10) is obtained by adding the obtained four solutions u_1, u_2, u_3 and u_4.

Example 1. Find the solution of the boundary value problem that satisfies $\Delta u = 0$ in $R = \{(x, y) : 0 < x < \pi \ , \ \ 0 < y < \pi\}$ and satisfies the conditions $u(0, y) = 0$, $u(\pi, y) = 0$, $u(x, 0) = 0$, $u(x, \pi) = \sin^3 x$.

Solution: By trigonometric identities, we can write

$$f(x) = \sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x.$$

From (8), the solution to the problem is found

$$u(x, y) = \sum_{n=1}^{\infty} b_n \frac{\sinh ny}{\sinh n\pi} \ \sin nx$$

and the coefficients b_n are obtained by

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \quad ; \qquad n = 1, 2, ...$$

or

$$f(x) = \sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_1 = \frac{3}{4} \quad , \qquad b_3 = -\frac{1}{4} \quad ; \qquad b_n = 0 \quad \text{(for all other } n\text{)}$$

Thus, the desired solution is found as

$$u(x, y) = \frac{3}{4} \frac{\sinh y}{\sinh \pi} \sin x - \frac{1}{4} \frac{\sinh 3y}{\sinh 3\pi} \sin 3x.$$