4.3. Dirichlet Problem for a Circular Disk

Let D denote the interior of the circle $x^2+y^2 \leq a^2$ in the xy-plane and C denote boundary of the circle

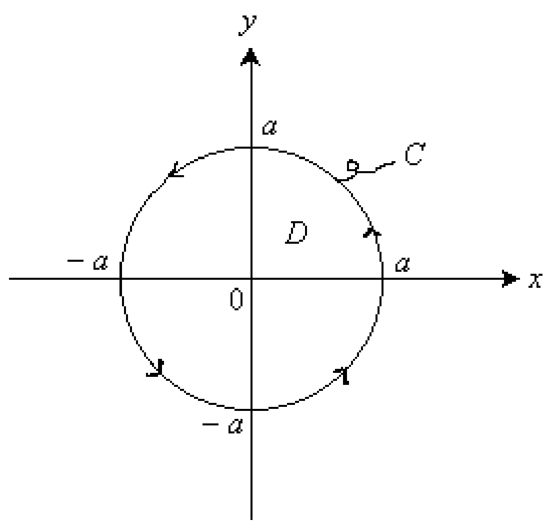

Figure 4.3. Dirichlet problem for a circular disk

We want to find a function $u(x,y)$ that is harmonic and equal to a previously given function $f(x,y)$ on the boundary C. That is, we want to solve the Dirichlet problem given as

$$\left.\begin{array}{lll} u_{xx}+u_{yy}=0 & , & \text{Inside } D \\ u=f^*(x,y) & , & \text{on } C \end{array}\right\} \tag{1}$$

In order to apply the method of separation of variables to such a problem, let's first write it in polar coordinates

$$x=r\cos\theta \quad , \quad y=r\sin\theta \quad ; \quad 0\leq\theta\leq 2\pi \quad , \quad 0<r\leq a \ .$$

Thus, we obtain the new form of the Laplace equation in (1) as follows

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0 \tag{2}$$

In this case, the boundary condition (1) will be transformed as

$$u(a,\theta)=f(\theta) \quad ; \quad 0\leq\theta\leq 2\pi \tag{3}$$

where the function $f(\theta)$ is defined as

$$f(\theta)=f^*(a\cos\theta,a\sin\theta).$$

Thus, the boundary value problem given by (1) turns into another boundary value problem that is equivalent to (2) and (3). Now let's apply the method of separation of variables to this problem and assume the existence of a following solution

$$u(r,\theta) = R(r)\,\Theta(\theta).$$

If the necessary derivatives are taken and they are put in their places at (2), we obtain

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda$$

where λ is the separation constant. This last equation takes us to the following ordinary differential equations that functions R and Θ, respectively, satisfy

$$r^2R''(r) + rR'(r) - \lambda R(r) = 0 \tag{4}$$

and

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \tag{5}$$

respectively. In order for the solution of our problem to be a single-valued function, its solution must be a 2π period function.That is,

$$u(r,\theta+2\pi) = u(r,\theta)$$

must be provided. If Θ is 2π periodic, then the requirement is satisfied. Thus Θ must satisfy the following periodic boundary conditions

$$\Theta(-\pi) = \Theta(\pi) \qquad , \qquad \Theta^{'}(-\pi) = \Theta^{'}(\pi). \tag{6}$$

With the condition (6), the eigenvalues of the Sturm Liouville system given by (5) are obtained as

$$\lambda_n = n^2 \quad ; \quad n = 0,1,2,...$$

and the corresponding eigenfunctions are written as

$$\Theta_n(\theta) \ = C_n\cos n\theta + D_n\sin n\theta.$$

For $\lambda = 0,$ the general solution of equation (4) should be

$$R_0(r) = A_0 + B_0\ln r$$

and for $\lambda_n = n^2 \quad ; \quad n = 0,1,2,...,$ the general solution of the equation can be found as

$$R_n(r) = A_nr^n + B_nr^{-n}$$

Thus, all functions in the form of $u_n(r,\theta)$ given by

$$u_n(r,\theta) = A_0 + B_0\ln r + (A_nr^n + B_nr^{-n})(C_n\cos n\theta + D_n\sin n\theta) \qquad (n = 1,2,... \qquad r > 0) \tag{7}$$

are periodic functions of 2π period and they satisfy the Laplace equation in polar coordinates (2). These functions are called *circular harmonics.*

In order to obtain a solution to our problem given by (2) and (3), we will construct a linear combination with infinite terms of functions (7). Since such a solution must be continuous in a region D that includes the origin point $r = 0$, we must make assumptions to eliminate the logarithmic term and the terms containing the negative powers of r. This means that in $(7), B_n = 0$ should be chosen for $n = 0, 1, 2, ...$and for the solution, a series of the following form should be considered

$$u(r, \theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} r^n (\alpha_n \cos n\theta + \beta_n \sin n\theta) \tag{8}$$

where we use the usual notation $\alpha_0, \alpha_n, \beta_n$ satisfying

$$A_0 = \frac{\alpha_0}{2} \quad , \quad A_n C_n = \alpha_n \quad , \quad A_n D_n = \beta_n.$$

Now let's apply the boundary condition (3) to determine these constants. In this case,

$$f(\theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} a^n (\alpha_n \cos n\theta + \beta_n \sin n\theta) \tag{9}$$

is obtained. This expression is the Fourier series of f in the interval $[-\pi, \pi]$ and the Fourier coefficients are expressed as follows

$$\left.\begin{array}{lll} a_n = a^n \alpha_n = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta & ; & n = 0, 1, 2, ... \\ b_n = a^n \beta_n = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta & ; & n = 0, 1, 2, ... \end{array}\right\}. \tag{10}$$

If these values of α_n and β_n are put in their places (8),

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (\frac{r}{a})^n (a_n \cos n\theta + b_n \sin n\theta) \tag{11}$$

is obtained. Here a_n and b_n are the Fourier coefficients of f in $[-\pi, \pi]$ and are given by integrals (10). The formula (11) together with the coefficients (10) gives the solution we are looking for, namely the solution of the Dirichlet problem for a circle given by (2) and (3). Now let's show that indeed the series (11) gives the solution to the Dirichlet problem in question. For this, let us first assume that the function f is a continuous piecewise differentiable and periodic function with a period of 2π in $[-\pi, \pi]$ and we say

$$M = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} |f(\theta)| \, d\theta.$$

It is clear from (10) that $|a_n| \leq M$ and $|b_n| \leq M$. On the other hand, for

$$u_0 = \frac{a_0}{2}$$

if we define $u_n(r,\theta)$ as follows

$$u_n(r,\theta) = (\frac{r}{a})^n (a_n \cos n\theta + b_n \sin n\theta) \quad ; \qquad n = 1, 2, ...$$

for any $r_0 < a$ and $r \le r_0$, we have

$$|u_n(r,\theta)| \le (\frac{r}{a})^n \left[|a_n| + |b_n|\right] \le 2M(\frac{r}{a})^n \tag{12}$$

Since the series with the general term $2M(\frac{r}{a})^n$ is uniformly convergent for $r \le r_0$, the series (11) will converge uniformly to $u(r,\theta)$ $0 \le r \le a$ and so the function $u(r,\theta)$ will be continuous for $0 \le r \le a$, $0 \le \theta \le 2\pi$. Therefore, the boundary condition (3) is satisfied for $r = a$

$$u(a,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) = f(\theta).$$

Now let's show that u satisfies the Laplace equation (2). If we take derivatives term by term from the expression (11) of $u(r,\theta)$ defined by a uniformly convergent series and write in the Laplace equation (2), we have

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \sum_{n=1}^{\infty} \frac{1}{r^2} u_n(r,\theta) \left[n(n-1) + n - n^2\right] = 0,$$

so u verifies the Laplace equation.

Example 2. Solve the Dirichlet problem given as

$$\begin{array}{lcl} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & ; & 0 < r < a \\ u(a,\theta) = a\cos^2\theta & ; & 0 \le \theta \le 2\pi \end{array}$$

Solution: Since $a\cos^2\theta = a(\frac{1}{2} + \frac{1}{2}\cos 2\theta)$, the desired solution is obtained by using the formula (11)

$$u(r,\theta) = \frac{1}{2}(a + \frac{r^2}{a}\cos 2\theta).$$

Example 3. Find the solution for the Dirichlet problem given by

$$\begin{array}{lcl} u_{xx} + u_{yy} = 0 & ; & \text{for } \ x^2 + y^2 < 1 \\ u = y^2 & ; & \text{for } \ x^2 + y^2 = 1 \end{array}.$$

Solution: Let's write the problem in polar coordinates. Using these coordinates

$$x = r\cos\theta \quad , \qquad y = r\sin\theta,$$

the above problem becomes as follows

$$\begin{array}{lll} u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0 & ; & r<1 \\ u(1,\theta)=\left.(r\sin\theta)^2\right|_{r=1}=\sin^2\theta=\dfrac{1}{2}(1-\cos 2\theta) & ; & 0\le\theta\le 2\pi \end{array}.$$

Thus using formula

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(\frac{r}{a})^n(a_n\cos n\theta+b_n\sin n\theta)$$

we obtain the solution in polar coordinates

$$u(r,\theta)=\frac{1}{2}(1-r^2\cos 2\theta).$$

If we return to cartesian coordinate system

$$\begin{aligned} u(r,\theta) &= \frac{1}{2}\left[1-r^2(1-2\sin^2\theta)\right]=\frac{1}{2}\left[1-r^2+2(r\sin\theta)^2\right] \\ &= \frac{1}{2}\left[1-(x^2+y^2)+2y^2\right], \end{aligned}$$

we find

$$u(x,y)=\frac{1}{2}\left[1-x^2+y^2\right].$$

4.4. Properties of Laplace Equation

In this section we give some qualitative properties that may be derived for Laplace's equation.

Mean value theorem

The solution of Laplace's equation inside a circle, obtained in the previous section by the method of separation of variables, gives an important result. If we evaluate the temperature at the origin, $r=0$, we discover from (8) in section 4.3. that

$$u(0,\theta)=a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta)\,d\theta,$$

the temperature there equals the average value of the temperature at the edges of the circle. This is called the **mean value property** for Laplace's equation. It is satisfied in general in the following specific sense. Suppose that we solve Laplace's equation in any region D. Let take any point Q inside D and a circle of any radius ρ (such that the circle is inside D). Let the temperature on the circle be $f(\theta)$, using polar coordinates centered at Q. The temperature at any point is the average of the temperature along any circle of radius ρ (lying inside D) centered at that point.

Maximum principles

Suppose that the function u is solution of the Laplace equation in bounded region D. Let u be continuous function which is not equal a constant in $\overline{D}$. Then, it takes the maximum and minimum values on the boundary of the region.

The maximum principle for Laplace's equation is proved by Mean value theorem.

Well-posedness and uniqueness

The maximum principle is a very useful tool for further analysis of partial differential equations, especially in establishing qualitative properties. We say that a problem is well posed if there exists a unique solution that depends continuously on the nonhomogeneous data (i.e., the solution varies a small amount if the data are slightly changed). This is an important tool for physical problems. If the solution changed dramatically with only a small change in the data, then any physical measurement would have to be exact in order for the solution to be reliable. Most standard problems in partial differential equations are well posed. For example, the maximum principle is used to prove that Laplace's equation $\Delta u = 0$ with $u = f(x)$ on the boundary is well-posed.

Suppose that we change the boundary data a small amount such that $\Delta u = 0$ with $v = h(x)$ on the boundary, where $h(x)$ is nearly the same as $f(x)$ everywhere on the boundary. We say $z = u - v$. From the linearity property,

$$\Delta z = \Delta u - \Delta v = 0$$

on the boundary. From the maximum (and minimum) principles for Laplace's equation, the maximum and minimum occur on the boundary. Thus, at any point inside,

$$\min(f(x) - h(x)) \leq z \leq \max(f(x) - h(x))$$

Due to the fact that $h(x)$ is nearly the same as $f(x)$ everywhere, z is small, and thus the solution v is nearly the same as u; the solution of Laplace's equation changes a small amount when the boundary data change slightly.

On the other hand, we can also prove that the solution of Laplace's equation is unique. Suppose that there are two solutions u and v satisfying Laplace's equation that verify the same boundary condition $f(x) = h(x)$. From the maximum and minimum principle, for the difference $z = u - v$ it holds

$$0 \leq z \leq 0$$

inside the region. So $z = 0$ everywhere inside, and thus $u = v$, which shows that if a solution exists, it must be unique. Since the properties (uniqueness and continuous dependence on the data) are satisfied, Laplace's equation with u specified on the boundary is a well-posed problem.