6. Sturm-Liouville Eigenvalue Problems

6.1. Some Examples

Heat Flow in a Nonuniform Rod

The temperature u in a nonuniform rod solves the partial differential equation

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q \tag{1}$$

where Q denotes any possible sources of heat energy. The thermal coefficient c, ρ and K_0 depend on x. The method of separation of variables is applied if (1) is linear and homogeneous. Usually, we consider the case $Q = 0$. But, we will be slightly more general. Assume that the heat source Q is proportunal to u,

$$Q = au$$

where a can depend on x (but not on t). So

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + au. \tag{2}$$

We apply the method of separation of variables to solve this equation. We assume that there is a homogeneous boundary condition (unspecified) at end points $x = 0$ and $x = L$.

Consider

$$u(x,t) = X(x)T(t). \tag{3}$$

If we substitute this function in (2), we obtain

$$c\rho X(x)\frac{dT}{dt} = T(t)\frac{d}{dx}\left(K_0\frac{dX}{dx}\right) + aX(x)T(t).$$

If we divide by $c\rho X(x)T(t)$, we have

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{c\rho X}\frac{d}{dx}\left(K_0\frac{dX}{dx}\right) + \frac{a}{c\rho} = -\lambda \tag{4}$$

where -λ is separation constant.

$$\begin{aligned}\frac{dT}{dt} &= -\lambda T \\ &\Rightarrow T(t) = Ce^{-\lambda t}\end{aligned}$$

It is seen that if $\lambda > 0$, it has exponentially decaying solutions; if $\lambda < 0$, solution grows and if $\lambda = 0$ solution is constant.

The spatial differential equation is

$$\frac{d}{dx}\left(K_0\frac{dX}{dx}\right)+aX+\lambda c\rho X=0. \tag{5}$$

If two homogeneous boundary conditions are given, it forms a boundary value problem. Here, the thermal coefficients a, c, ρ, K_0 are not constant and the equation (5) is a differential equation with nonconstant-coefficient. In general, nonconstant-coefficient differential equations occur appear in situations where physical properties are nonuniform. Generally, we can not solve (5) in the variable-coefficient case, but we can find a numerical approximate solution on the computer. Later, we will return to reinvestigate heat flow in a nonuniform rod.

Circularly Symmetric Heat Flow

The differential equations with nonconstant-coefficient also arise if the physical parameters are constant. In Section 1.5 we showed that if the temperature u in some plane two dimensional region is circularly symmetric, that is, u depends only on time t and on the radial distance r from the origin), then u is the solution of the linear and homogeneous partial differential equation

$$\frac{\partial u}{\partial t}=k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \tag{6}$$

where we assume that all the thermal coefficients are constant. By the method of separation of variables, we seek for a solution in the form

$$u(r,t)=R(r)\,T(t).$$

After necessary calculations, we obtain

$$\frac{1}{kT(t)}\frac{dT(t)}{dt}=\frac{1}{rR(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right)=-\lambda,$$

which gives two differential equations

$$\frac{dT(t)}{dt}=-\lambda kT(t) \tag{7}$$

and

$$\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right)+\lambda rR(r)=0. \tag{8}$$

Here, the separation constant is denoted -λ, because we expect solutions to exponentially decay in time when $\lambda > 0$. The solution of (7) is $T(t)=Ce^{-\lambda kt}$ and also the equation (8) will be solved in terms of Bessel functions later.

We now consider the appropriate homogeneous boundary conditions for circularly symmetric heat flow inside circle and a circular annulus: In both cases, let all boundaries be fixed at zero temperature. For the annulus, the boundary conditions for (8) at the inner ($r=a$) and outer ($r=b$) concentric circular walls

$$u(a,t)=0 \text{ and } u(b,t)=0,$$

for the circle, the boundary condition for (8) is $u(b,t)=0$. Because of the fact that the physical variable r ranges from 0 to b, we need a homogeneous boundary condition at $r=0$ for mathematical reasons. So, we expect u bounded at $r=0$, that is, $|u(0,t)|<\infty$. Thus, we have homogeneous conditions at both $r=0$ and $r=b$ for the circle.

6.2. General Classification

A boundary value problem is formed of a linear homogeneous differential equation and corresponding linear homogeneous boundary conditions. All of the differential equations for boundary value problems are in form of

$$\frac{d}{dx}\left(p\frac{d\phi}{dx}\right)+q\phi+\lambda\sigma\phi=0 \quad , \quad a<x<b \tag{9}$$

here λ denotes eigenvalue. Some examples of (9) are as follows:

a) Simplest case: $\frac{d^2\phi}{dx^2}+\lambda\phi=0$ for $p=1, q=0$, $\sigma=1$.

b) Heat flow in a nonuniform rod: $\frac{d}{dx}\left(K_0\frac{dX}{dx}\right)+aX+\lambda c\rho X=0$ here the dependent variable $\phi=X$ and $p=K_0, q=a, \sigma=c\rho$.

c) Circularly symmetric heat flow: $\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\lambda rR=0$ here the dependent variable $\phi=R$, the independent variable $x=r$ and $p(x)=x, q(x)=0$, $\sigma(x)=x$.

Many interesting results are related to any equation in the form (9). This equation is called a Sturm–Liouville differential equation.

Boundary conditions. Some linear homogeneous boundary conditions are as follows:

	Heat flow	**Mathematical terminology**
$\phi=0$	Fixed(zero) temperature	Dirichlet condition
$\frac{d\phi}{dx}=0$	Insulated	Neumann condition
$\frac{d\phi}{dx}=\mp h\phi$	(Homogeneous) Newton's law of cooling 0^0 temperature, $h=H/K_0, h>0$	Robin condition
$\phi(-L)=\phi(L)$ $\frac{d\phi}{dx}(-L)=\frac{d\phi}{dx}(L)$	Perfect thermal contact	Periodicity condition (mixed type)
$\lvert\phi(0)\rvert<\infty$	Bounded temperature	Singularity condition