6.4. Problems with a boundary condition of the third type

In this section we consider problems with a boundary condition of the third kind with constant physical parameters.

While heat flow in a uniform rod satisfies

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

a uniform vibrating string verifies

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}. \tag{2}$$

We suppose that the left end is fixed, but the right end satisfies a homogeneous boundary condition of the third kind:

$$u(0,t) = 0, \tag{3}$$

$$\frac{\partial u}{\partial x}(L,t) = -hu(L,t) \tag{4}$$

We note that for heat conduction, the condition (4) corresponds to Newton's law of cooling if $h > 0$, and for the vibrating string problem, (4) corresponds to a restoring force if $h > 0$, the so-called elastic boundary condition. In physical problems, usually $h \geq 0$. But for mathematical results, we will study both cases $h \geq 0$ and $h < 0$.

By the method of separation of variables, we seek for a solution in the form

$$u(x,t) = T(t)\,X(x), \tag{5}$$

the time part verifies the following differential equation

$$\text{heat flow} \quad : \quad \frac{dT}{dt} = -\lambda kT \tag{6}$$

$$\text{vibrating string} \quad : \quad \frac{d^2T}{dt^2} = -\lambda c^2 T \tag{7}$$

The spatial part, $X(x)$, verifies the following regular Sturm–Liouville eigenvalue problem:

$$\frac{d^2X}{dx^2} + \lambda X = 0 \tag{8}$$

$$X(0) = 0 \tag{9}$$

$$\frac{dX}{dx}(L) + hX(L) = 0. \tag{10}$$

Here h is a given fixed constant. If $h \geq 0$, we call it the "physical" case, while if $h < 0$, we call it the "nonphysical" case.

When the regular Sturm–Liouville eigenvalue problem (8)-(10) is solved, there are five different cases depending on the value of the parameter h in the boundary condition.

In physical case, there are two cases. In the nonphysical case, there are only three cases: If $-1 < hL$, all the eigenvalues are positive; if $hL = -1$, there are no negative eigenvalues, but zero is an eigenvalue; and if $hL < -1$, there are still an infinite number of positive eigenvalues, but there is also one negative one.

For these cases, the eigenvalues and corresponding eigenfunctions are given below.

Case I. Assume that $h > 0$. If we solve the equation (8), we find

$$X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x.$$

If we apply the boundary condition $X(0) = 0$, we find $c_1 = 0$ and then we have

$$X(x) = c_2 \sin\sqrt{\lambda}x.$$

By differentiating this function, we obtain

$$X'(x) = c_2\sqrt{\lambda}\cos\sqrt{\lambda}x. \tag{11}$$

The boundary condition of the third kind implies that

$$c_2\left(\sqrt{\lambda}\cos\sqrt{\lambda}L + h\sin\sqrt{\lambda}L\right) = 0.$$

For $c_2 \neq 0$, there exists eigenvalues λ for $\lambda > 0$ such that these eigenvalues satisfy

$$\sqrt{\lambda}\cos\sqrt{\lambda}L + h\sin\sqrt{\lambda}L = 0.$$

We can not determine these eigenvalues exactly. But, they can be determined graphically. The eigenfunctions are

$$X(x) = \sin\sqrt{\lambda}x.$$

Case II. Assume that $h = 0$. In the equation (11), we apply the condition

$$\frac{dX}{dx}(L) = 0.$$

So, we obtain

$$\begin{aligned} X'(L) &= c_2\sqrt{\lambda}\cos\sqrt{\lambda}L = 0 \\ &\Rightarrow \sqrt{\lambda}\cos\sqrt{\lambda}L = 0 \quad (c_2 \neq 0) \end{aligned}$$

For $\lambda > 0$, we have eigenvalues

$$\cos\sqrt{\lambda}L = 0 \Rightarrow \lambda = \left(\frac{(n-1/2)\pi}{L}\right)^2 \quad , \quad n = 1, 2, ...$$

The eigenfunctions are $\sin\sqrt{\lambda}x$.

Similarly, in the nonphysical case, there are only three cases: when $-1 < hL < 0$, for $\lambda > 0$ the eigenfunctions are $\sin\sqrt{\lambda}x$. When $hL = -1$, for $\lambda > 0$ the eigenfunctions are $\sin\sqrt{\lambda}x$ and for $\lambda = 0$ the eigenfunction is x. When $hL < -1$, for $\lambda > 0$ the eigenfunctions are $\sin\sqrt{\lambda}x$ and for $\lambda < 0$ the eigenfunctions are $\sinh\sqrt{s_1}x$ (here $s_1 = -\lambda$).