Section 2. Partial Differential Equations of the First Order

2.1. Partial Differential Equations

Partial differential equations arise in geometry and physics when the number of independent variables in the problem is two or more. In a such case, any dependent variable is a function of more than one variable so that it possesses partial derivatives with respect to several variables.

Consider a relation between the derivatives in the form

$$F\left(\frac{\partial\theta}{\partial x},...,\frac{\partial^2\theta}{\partial x^2},...,\frac{\partial^2\theta}{\partial x\partial t},...\right)=0. \tag{1}$$

Such an equation is called a 'partial differential equation'. We define the order of a partial differential equation to be the order of the derivative of highest order in the equation. For example, let θ be the dependent variable and x, y, and t be independent variables. Then, the equation

$$\frac{\partial^2\theta}{\partial x^2}=\frac{\partial\theta}{\partial t} \tag{2}$$

is a second-order equation in two variables, the equation

$$\left(\frac{\partial\theta}{\partial x}\right)^3+\frac{\partial\theta}{\partial t}=0 \tag{3}$$

is a first-order equation in two variables. The equation

$$x\frac{\partial\theta}{\partial x}+y\frac{\partial\theta}{\partial y}+\frac{\partial\theta}{\partial t}=0 \tag{4}$$

is a first-order equation in three variables.

In this section, we consider partial differential equations of the first order, i.e., equations of the type

$$F\left(\theta,\frac{\partial\theta}{\partial x},...\right)=0 \tag{5}$$

We suppose that there are two independent variables x and y and suppose that the dependent variable is z. If we write

$$p=\frac{\partial z}{\partial x}\ ,\ q=\frac{\partial z}{\partial y}. \tag{6}$$

Such an equation can be written in the form

$$f\left(x,y,z,p,q\right)=0 \tag{7}$$

2.2.Origins of first-order partial differential equations

Before studying the solution of equations of the type (7), we explain how they arise. Consider the equation

$$x^2+y^2+\left(z-b\right)^2=a^2 \tag{8}$$

where a and b are arbitrary constants. The equation (8) gives the set of all spheres with center $(0,0,b)$. Differentiating this equation with respect to x, we have

$$\begin{aligned} &\Rightarrow \quad 2x + 2\,(z-b)\,z_x = 0 \\ &\Rightarrow \quad x + (z-b)\,p = 0. \end{aligned}$$

If we differentiate the equation (8) with respect to y, we obtain

$$\begin{aligned} &\Rightarrow \quad 2y + 2\,(z-b)\,z_y = 0 \\ &\Rightarrow \quad y + (z-b)\,q = 0. \end{aligned}$$

From these two equations if we eliminate the constant b, we find the partial differential equation

$$yp - xq = 0 \tag{9}$$

which is of the first order. The set of all spheres with centers $(0,0,b)$ on the z-axis is characterized by the partial differential equation (9). On the other hand, other geometrical entities can be described by the same equation (9). For example, consider the equation

$$x^2 + y^2 = (z-c)^2 \tan^2 \alpha \tag{10}$$

where c and α are arbitrary constants and it represents the set of all right circular cones whose axes coincide with the line $0z$. If we differentiate the equation (10) with respect to x and y, respectively we obtain

$$p\,(z-c)\tan^2\alpha = x \quad , \quad q\,(z-c)\tan^2\alpha = y \tag{11}$$

If we eliminate c and α from these equations, we find equation (9) for these cones.

We note that what the sphere and cones have in common is that they are surfaces of revolution which have the line $0z$ as axes of symmetry.

With this property, all surfaces of revolution are characterized by the equation

$$z = f\left(x^2 + y^2\right) \tag{12}$$

where the function f is arbitrary. If we differentiate equation (12) with respect to x and y, respectively, we have

$$p = 2xf'\left(x^2+y^2\right) \quad , \quad q = 2yf'\left(x^2+y^2\right).$$

By eliminating f, we find equation (9).

Thus, it is seen that the function z defined by the equations (8), (10) and (12) is a 'solution' of the equation (9).

Now, we generalize this argument. The relations (8) and (10) are of type

$$F\,(x,y,z,a,b) = 0 \tag{13}$$

in which a and b are arbitrary constants. If we differentiate this equation with respect to x and y, respectively, we have

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0 \quad , \quad \frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0. \tag{14}$$

From (13) and (14), between these equations involving arbitrary constants a and b, we can eliminate a and b. Then, we obtain a relation in the form

$$f\left(x, y, z, p, q\right) = 0 \tag{15}$$

It shows that the system of surfaces (13) gives rise to a partial differential equation (15) of the first order. The obvious generalization of the relation (12) is a relation $x, y,$ and z in the form

$$F\left(u, v\right) = 0 \tag{16}$$

where u and v are known functions of $x, y,$ and z and F is an arbitrary function of u and v.Differentiating the equation (16) with respect to x and y, respectively, we obtain

$$\begin{aligned}
\frac{\partial F}{\partial u}\left\{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right\} + \frac{\partial F}{\partial v}\left\{\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right\} &= 0 \\
\frac{\partial F}{\partial u}\left\{\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right\} + \frac{\partial F}{\partial v}\left\{\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right\} &= 0.
\end{aligned}$$

If we eliminate $\dfrac{\partial F}{\partial u}$ and $\dfrac{\partial F}{\partial v}$ from these equations, we find that

$$p\frac{\partial\left(u, v\right)}{\partial\left(y, z\right)} + q\frac{\partial\left(u, v\right)}{\partial\left(z, x\right)} = \frac{\partial\left(u, v\right)}{\partial\left(x, y\right)}, \tag{17}$$

which is partial differential equation of the form (15). This equation is 'linear equation'; i.e., the powers p and q are both unity but (15) does not need to be linear.

$\Longrightarrow$ A first order quasi-linear PDE must be of the form

$$P\left(x, y, z\right) z_x + Q\left(x, y, z\right) z_y = R\left(x, y, z\right)$$

$\Longrightarrow$ A first order quasi-linear PDE where P, Q are functions of x and y alone is a semi-linear PDE.

$$P\left(x, y\right) z_x + Q\left(x, y\right) z_y = R\left(x, y, z\right)$$

$\Longrightarrow$ A first order semi-linear PDE where $R\left(x, y, z\right) = R_0\left(x, y\right) z + R_1\left(x, y\right)$ is a linear PDE.

$$P(x,y)\,z_x + Q(x,y)\,z_y = R_0(x,y)\,z + R_1(x,y)$$

$\Longrightarrow$ A PDE which is not quasi-linear is called nonlinear PDE.

As an example, the equation

$$(x-a)^2 + (y-b)^2 + z^2 = 1$$

which is the set of all spheres of unit radius with center $(a,b,0)$ leads to the first order nonlinear partial differential equation

$$z^2\left(1+p^2+q^2\right) = 1.$$