2.5. Nonlinear Partial Differential Equations of the First Order

The partial differential equations of the first order, independent variables x and y and the dependent variable z, are most generally expressed as

$$F(x, y, z, p, q) = 0. \tag{1}$$

In the special case that the function F is linear with respect to the partial derivatives $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$, the equation (1) reduce to the linear and quasi-linear partial differential equations that we examined in the previous section. In the general case that we will examine in this section, the function F does not have to be linear with respect to p and q. As we will recall from Section 1, the partial differential equation of a two-parameter family defined by

$$f(x, y, z, a, b) = 0 \tag{2}$$

is given by the equation (1). The reverse of this expression is also true. That is, any partial differential equation of type (1) has solutions of type (2). It will be given later with the Charpit method. Three types of integrals corresponding to solutions of equations of type (1) are defined as follows.

A) Any two-parameter solution (2) of equation (1) is called a complete integral of the equation.

B) If there exists a functional relation such as

$$b = \phi(a) \tag{3}$$

between arbitrary constants a and b in (2), then an one-parameter family is defined as

$$f(x, y, z, a, \phi(a)) = 0, \tag{4}$$

which is a subfamily of (2). In (4), a different subfamily is usually obtained for each different choice of the function ϕ. If there exists any surface tangent to each member of a family of type (4), such a surface is called the envelope of family (4) and the equation for this envelope surface is obtained by eliminating the parameter a between the equations

$$f(x, y, z, a, \phi(a)) = 0 \quad , \qquad \frac{\partial}{\partial a} f(x, y, z, a, \phi(a)) = 0. \tag{5}$$

For all possible choices of ϕ, such envelope surfaces to be obtained from equations (5) are also solutions of equation (1) and they are known as the *general integral* of the partial differential equation (1). A special case of the general integral for a certain function $\phi(a)$ is found. If ϕ is taken as an arbitrary function, it is not always possible to eliminate the constant a between equations (5), so it is generally not possible to express the general integral of a nonlinear partial differential equation of first order with the help of an arbitrary function.

Also note that complete integrals are not a special case of a general integral. For each particular choice of a and ϕ, equations (5) define two special surfaces. When the two equations of (5) are taken together, a space curve, which is the intersection of these surfaces, is defined. This type of curve is called the *characteristic curve* of equation (1). For any particular choice of ϕ, the general integral surface to be obtained from (5) is tangent to each member of the family (4) along a characteristic curve. So, we can look at the general integral as the integral surfaces formed by the characteristic curves.

C) The two-parameter family given by (2) can also have an envelope. If there exists such an envelope, its equation will be obtained by eliminating the parameters a and b between the relations

$$f(x, y, z, a, b) = 0 \quad , \qquad \frac{\partial f}{\partial a} = 0 \quad , \qquad \frac{\partial f}{\partial b} = 0. \tag{6}$$

This envelope satisfies the equation (1) and it is called the *singular integral* of equation (1).

It should be noted that every function to be obtained from (6) does not have the feature of envelope. For this reason, it is necessary to check whether the obtained function satisfies the equation (1) or not. If equation (1) has any singular integral, this integral can also be obtained by eliminating p and q between equations

$$F(x, y, z, p, q) = 0 \quad , \qquad \frac{\partial F}{\partial p} = 0 \quad , \qquad \frac{\partial F}{\partial q} = 0.$$

Now let's explain these three types of solutions in the equation

$$(1 + p^2 + q^2)z^2 = 1. \tag{7}$$

It can be easily verified that a solution to this equation is given by

$$(x - a)^2 + (y - b)^2 + z^2 = 1 \tag{8}$$

where a and b are arbitrary constants. Since solution (8) contains two arbitrary constants, it is a complete integral of the equation (7). If $b = a$ in equation (8), then one has a one-parameter subfamily which is given below

$$(x - a)^2 + (y - a)^2 + z^2 = 1.$$

If the parameter a is eliminated from this equation and the equation

$$x + y - 2a = 0$$

which is obtained by taking a partial derivative with respect to a, the envelope is obtained as follows

$$(x - y)^2 + 2z^2 = 2. \tag{9}$$

Let's check if the equation (9), which is a cylinder equation, satisfies the partial differential equation (7). By differentiating both sides of the equation (9) with respect to x and y, respectively we can write

$$2zp = y - x \quad , \quad 2zq = x - y.$$

If both sides of these two equations are squared, summed up and divided by two, we have

$$(x - y)^2 = 2z^2(p^2 + q^2).$$

If this value of $(x - y)^2$ is substituted in (9), we obtain

$$2z^2(p^2 + q^2) + 2z^2 = 2 \ \Rightarrow \ (1 + p^2 + q^2)z^2 = 1,$$

which shows that (9) is an integral surface of equation (7). This solution is of the type described in (B). So, it is a general integral of equation (7). The envelope of the two-parameter family (8) is found by eliminating a and b between the equation (8) and the equations

$$x - a = 0 \ , \qquad y - b = 0$$

obtained by taking partial derivatives from (8) with respect to a and b. That is to say, the envelope consists of the pair of planes

$$z^2 = 1 \Rightarrow \quad z = 1 \quad \text{and} \quad z = -1.$$

It can be easily realized that these planes are integral surfaces of the equation (7). Since these integral surfaces are of type (C), they are the singular integrals of equation (7).

Before giving the Charpit method for solving a general equation of nonlinear of type (1), let's see the compatible systems on which this method is based.