2.7. Charpit Method

A method for solving the first order partial differential equation

$$F(x, y, z, p, q) = 0 \tag{1}$$

is given by Charpit. The basis of this method is based on finding a second equation which is given below

$$G(x, y, z, p, q, a) = 0 \tag{2}$$

which is compatible with equation (1) and contains an arbitrary constant a. The compatibility of equations (1) and (2) will require the identity

$$p\,(F_z G_p - G_z F_p) - q\,(F_q G_z - G_q F_z) - (F_q G_y - G_q F_y) + (F_x G_p - G_x F_p) \equiv 0 \tag{3}$$

which is given in the previous section. Considering that G is an unknown function, identity (3) can be written in another way.as follows

$$F_p \frac{\partial G}{\partial x} + F_q \frac{\partial G}{\partial y} + (pF_p + qF_q)\frac{\partial G}{\partial z} - (F_x + pF_z)\frac{\partial G}{\partial p} - (F_y + qF_z)\frac{\partial G}{\partial q} = 0. \tag{4}$$

This equation is a Lagrange linear equation in which x, y, z, p, q act as independent variables. Then, our problem is reduced to finding a first integral in the form of (2) from the auxiliary system

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q} = \frac{dp}{-(F_x + pF_z)} = \frac{dq}{-(F_y + qF_z)} \tag{5}$$

of the equation (4). It is not necessary to use all of the equations (5) for a first integral to be found from system (5), known as Charpit equations. However, in the first integral we will find, at least one of p or q must be found. Later, from the compatible system formed by equation (2) and equation (1),

$$p = p(x, y, z, a) \quad , \quad q = q(x, y, z, a)$$

are solved and

$$dz = p(x, y, z, a)\,dx + q(x, y, z, a)dy \tag{6}$$

is integrated. When equation (6) is integrated, a solution containing two arbitrary parameters

$$f(x, y, z, a, b) = 0$$

is obtained. This solution is a complete integral of equation (1).

Example 1. Find a complete integral of the equation $p^2x + q^2y = z$.

Solution: If we write the given equation as

$$F(x, y, z, p, q) = p^2x + q^2y - z = 0,$$

we have

$$F_x = p^2 \quad , \quad F_y = q^2 \quad , \quad F_z = -1 \quad , \quad F_p = 2px \quad , \quad F_q = 2qy.$$

The corresponding auxiliary system, that is, the Charpit equations are given as follows

$$\frac{dx}{2px} = \frac{dy}{2qy} = \frac{dz}{2(p^2x + q^2y)} = \frac{dp}{-p^2 + p} = \frac{dq}{-q^2 + q}.$$

Therefore, a first integral containing at least one of p and q can be obtained as follows:

$$\frac{p^2dx + 2pxdp}{2p^3x + 2px(p - p^2)} = \frac{q^2dy + 2qydq}{2q^3y + 2qy(q - q^2)} \quad , \quad \frac{d(p^2x)}{p^2x} = \frac{d(q^2y)}{q^2y}$$

by integrating both sides, we get

$$p^2x = aq^2y.$$

This equation in which a is arbitrary constant and the first equation form an compatible system. If we solve p and q from this system, we have

$$\left.\begin{array}{c} p^2x + q^2y = z \\ p^2x = aq^2y \end{array}\right\} \Rightarrow \qquad p = \left\{\frac{az}{(1+a)x}\right\}^{1/2} \quad , \quad q = \left\{\frac{z}{(1+a)y}\right\}^{1/2}$$

and If they are put in place in the equation $dz = pdx + qdy$, we obtain

$$dz = \left\{\frac{az}{(1+a)x}\right\}^{1/2} dx + \left\{\frac{z}{(1+a)y}\right\}^{1/2} dy$$

This expression is written as

$$\left(\frac{1+a}{z}\right)^{1/2} dz = \left(\frac{a}{x}\right)^{1/2} dx + \left(\frac{1}{y}\right)^{1/2} dy$$

and if both sides are integrated, we have

$$\sqrt{(1+a)z} = \sqrt{ax} + \sqrt{y} + b.$$

This is a complete integral of the given equation where a and b are parameters.

Example 2. Find a complete integral of the equation

$$pq + x(2y+1)p + (y^2 + y)q - (2y+1)z = 0$$

and obtain a singular integral, if any.

Solution: If the given equation is written as

$$F(x, y, z, p, q) = pq + x(2y+1)p + (y^2 + y)q - (2y+1)z = 0,$$

we have

$$F_x = (2y+1)p\,,\ F_y = 2xp + (2y+1)q - 2z\ ,\ F_z = -(2y+1)\,,$$

$$F_p = q + x(2y+1)\ ,\ F_q = p + y^2 + y$$

In the corresponding auxiliary system, since the denominator of dp is

$$F_x + pF_z = 0,$$

from $dp = 0$, $p = a$ is obtained as a first integral of the system. Substituting this value of $p = a$ into the given equation and solving q, we can write

$$q = \frac{(2y+1)(z-ax)}{y^2+y+a}.$$

From $dz = pdx + qdy$, we have

$$dz = adx + \frac{(2y+1)(z-ax)}{y^2+y+a}dy$$

or in other form, we obtain

$$\frac{dz - adx}{z - ax} = \frac{(2y+1)}{y^2+y+a}dy.$$

By integrating both sides, we find

$$\ln(z - ax) = \ln(y^2 + y + a) + \ln b,$$

from which it follows

$$z = ax + (y^2 + y + a)b.$$

Thus, the desired complete integral is obtained. If the derivatives are taken from this solution according to the parameters a and b, we have

$$x + b = 0\ \ ,\ \ y^2 + y + a = 0$$

When a and b are eliminated among the last three equations, we obtain

$$z = -x(y^2 + y).$$

This equation is the envelope of the family of two-parameter surface and is a singular integral of the given partial differential equation.