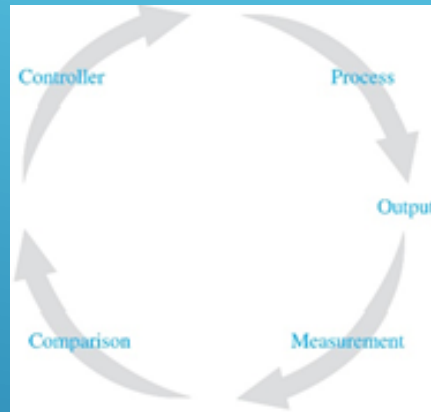


CONTROL SYSTEMS



Doç. Dr. Murat Efe

WEEK 10

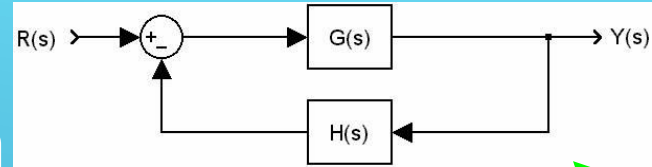
Root Locus Analysis - Summary

Rules for Constructing Root Loci

- 1. Locate the open loop poles and zeros**
- 2. Determine the loci on the real axis**
- 3. Determine the asymptotes of root loci**
- 4. Find the breakaway and break-in points**
- 5. Determine the angle of departure from a complex pole**
- 6. Determine the angle of arrival at a complex zero**
- 7. Find the point where the root loci may cross the imaginary axis**
- 8. Determine the shape of the root loci in the broad neighborhood of the $j\omega$ axis and the origin of the s-plane**
- 9. Determine the closed loop poles**

Root Locus Analysis

Pole-Zero Cancellation



$$G(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{D'(s)(s + \alpha)} \quad \text{and} \quad H(s) = \frac{A(s)}{B(s)} = \frac{A'(s)(s + \alpha)}{B(s)}$$

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{\frac{N(s)}{D'(s)(s + \alpha)}}{1 + \frac{N(s)}{D'(s)(s + \alpha)} \frac{A'(s)(s + \alpha)}{B(s)}}$$

$$= \frac{\frac{N(s)}{D'(s)(s + \alpha)} \frac{B(s)}{B(s)}}{\frac{D'(s)(s + \alpha) B(s)}{D'(s)B(s)} + \frac{N(s)A'(s)}{D'(s)B(s)}} = \frac{\frac{N(s)B(s)}{D'(s)B(s)}}{\frac{D'(s)B(s) + N(s)A'(s)}{D'(s)B(s)}} \frac{1}{(s + \alpha)}$$

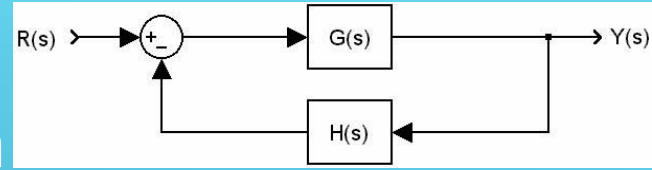
$$= \frac{N(s)B(s)}{D'(s)B(s) + N(s)A'(s)} \frac{1}{(s + \alpha)}$$

**Canceled pole of G(s)
is kept as a CL pole!**

Root Locus Analysis

Pole-Zero Cancellation

An Example



$$G(s) = \frac{s+1}{(s+2)(s+3)}, \text{ and } H(s) = K \frac{s+3}{s+4}$$

$$1 + KG(s)H(s) = 1 + \frac{K(s+1)}{(s+2)(s+4)} \left(\frac{s+3}{s+3} \right)$$

$$\left((s+2)(s+4) + K(s+1) \right) \underbrace{(s+3)}_{\text{circled}} = 0$$

$$\left((s+2)(s+4) + K(s+1) \right) = 0$$

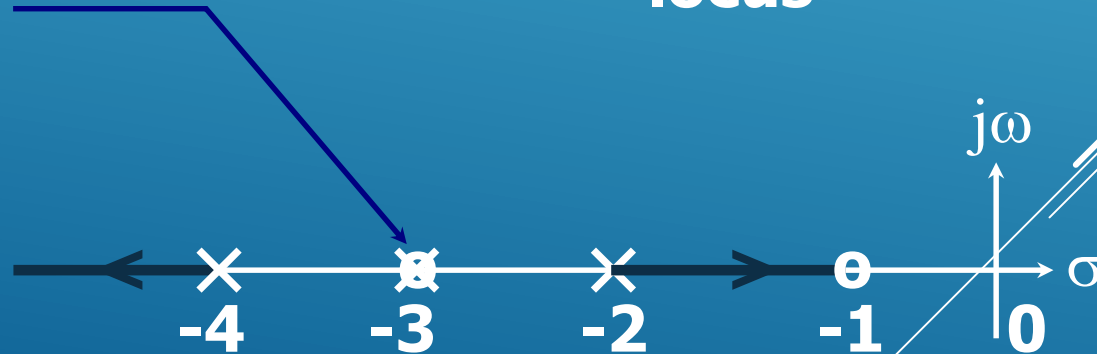
➡ **(s+3) is common**

➡ **(s+3) terms cancel**

➡ **Char. Eqn.**

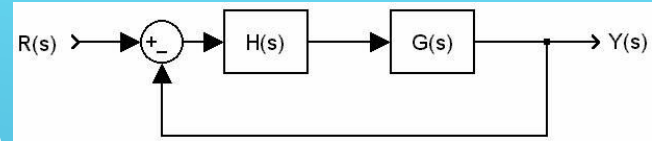
➡ **Char. Eqn. for root locus**

Here is the pole-zero cancellation!



Root Locus Analysis

Pole-Zero Cancellation



$$G(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{D'(s)(s + \alpha)} \quad \text{and} \quad H(s) = \frac{A(s)}{B(s)} = \frac{A'(s)(s + \alpha)}{B(s)}$$

$$T(s) = \frac{G(s)H(s)}{1 + G(s)H(s)} = \frac{\frac{N(s)}{D'(s)(s + \alpha)} \frac{A'(s)(s + \alpha)}{B(s)}}{1 + \frac{N(s)}{D'(s)(s + \alpha)} \frac{A'(s)(s + \alpha)}{B(s)}}$$

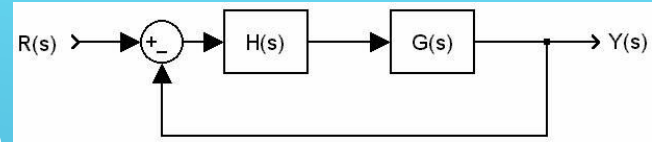
$$= \frac{\frac{N(s)A'(s)}{D'(s)B(s)}}{\frac{D'(s)B(s) + N(s)A'(s)}{D'(s)B(s)}} = \frac{N(s)A'(s)}{D'(s)B(s) + N(s)A'(s)}$$

Cancelled pole is not a closed loop pole at all



Root Locus Analysis Pole-Zero Cancellation

An Example (Same result is obtained!)



$$G(s) = \frac{s+1}{(s+2)(s+3)}, \text{ and } H(s) = K \frac{s+3}{s+4}$$

$$1 + KG(s)H(s) = 1 + \frac{K(s+1)}{(s+2)(s+4)} \left(\frac{s+3}{s+3} \right)$$

$$\left((s+2)(s+4) + K(s+1) \right) \underbrace{(s+3)}_{\text{circled}} = 0$$

$$\left((s+2)(s+4) + K(s+1) \right) = 0$$

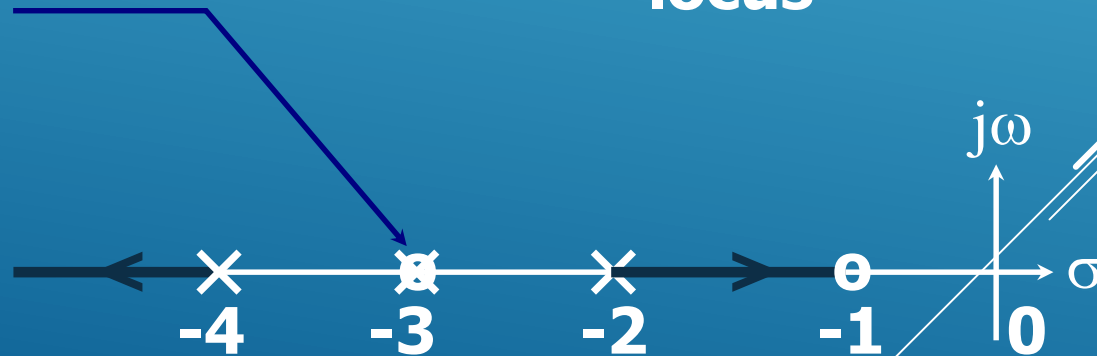
➡ **(s+3) is common**

➡ **(s+3) terms cancel**

➡ **Char. Eqn.**

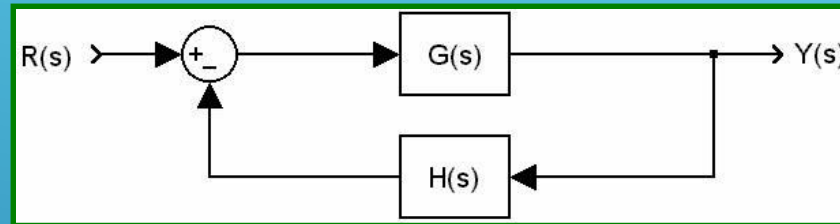
➡ **Char. Eqn. for root locus**

Here is the pole-zero cancellation!

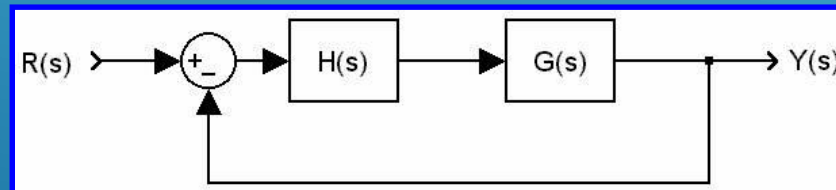


Root Locus Analysis

Pole-Zero Cancellation





Canceled pole is a CL pole
Root locus does not notice it



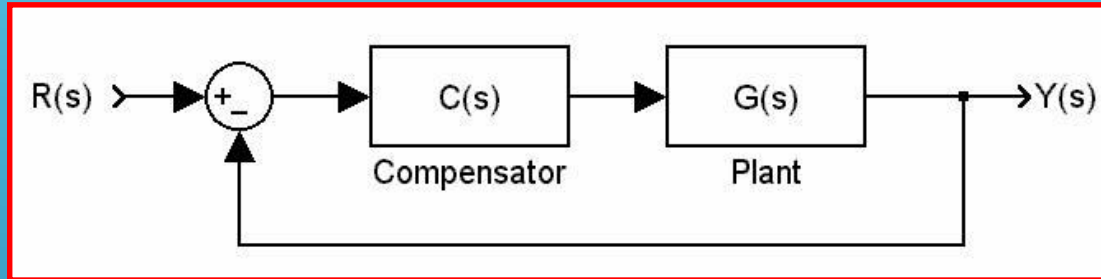
Canceled pole is NOT a CL pole
Root locus does not notice it



P-5 Design based on Root Locus

-  The goal is to meet the design specifications, and the way we followed so far has been to modify the gain K . What if this is not sufficient?
-  Modify the system dynamics suitably to obtain the desired result, which means **compensation**, and the device you used is called **compensator**.

Design based on Root Locus

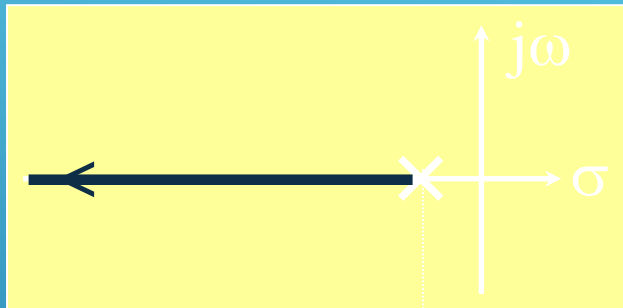
Description of the Compensation Problem



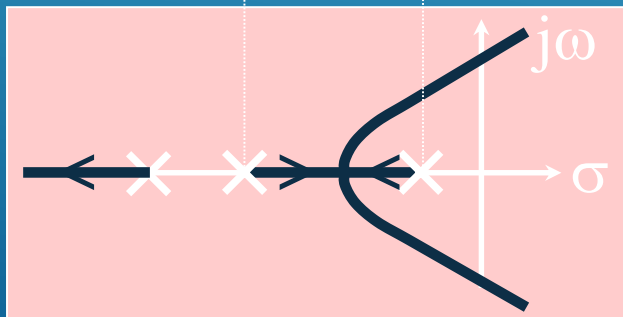
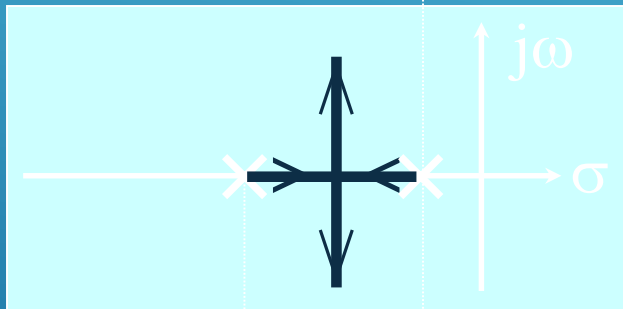
-  **C(s) may remove some poles of G(s) and may add new poles, or C(s) may remove some zeros of G(s) and may add new zeros to change the shape of root locus.**
-  **Once the shape of root locus becomes suitable to locate the desired closed loop poles, the adjustment of loop gain K is performed.**

Design based on Root Locus

Effects of Addition of Poles

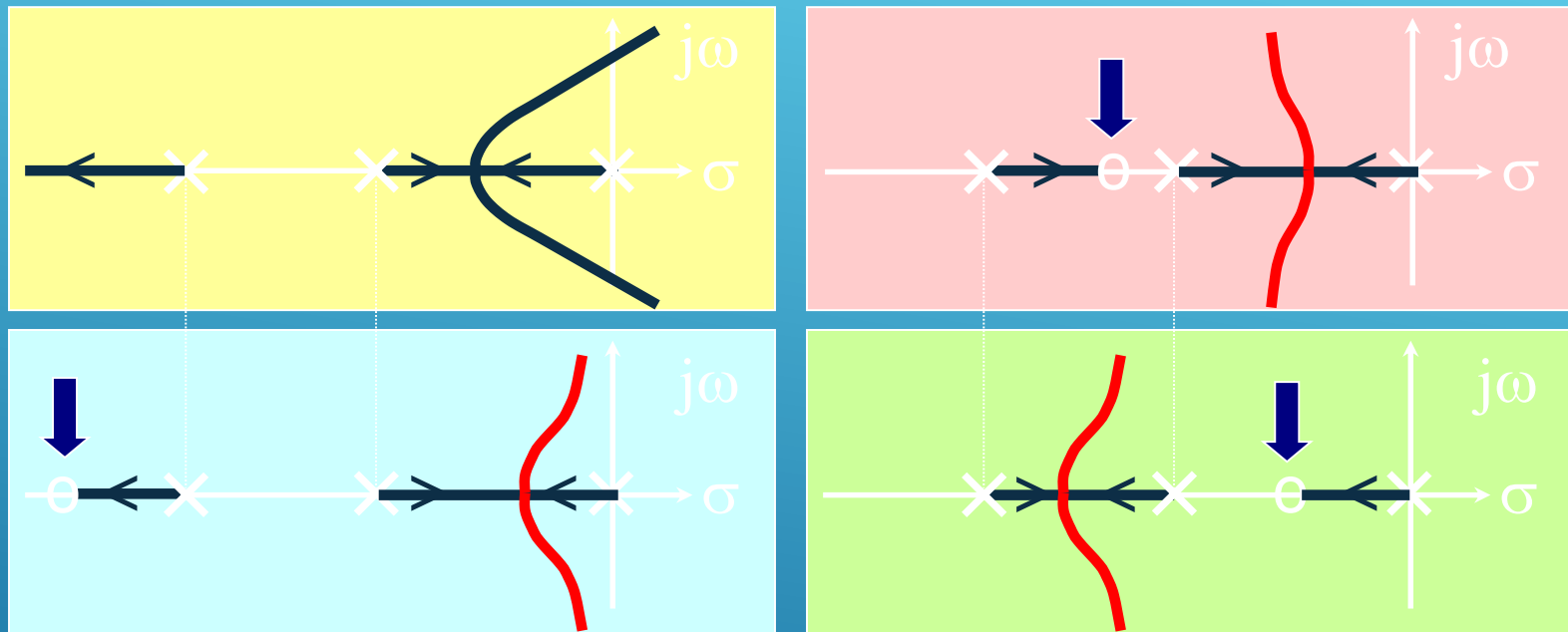


Adding poles pulls the root locus to the right



After some value of K, two of the CL poles are unstable!

Design based on Root Locus Effects of Addition of Zeros

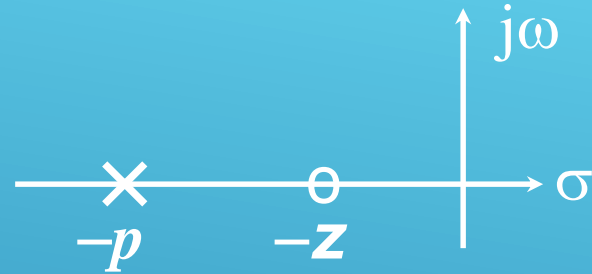


Adding zeros pulls the root locus to the left

Notice that, the CL poles are always stable for this example. Adding zeros increase the stability of the CL system, this is due to the anticipatory behavior of the derivative action.

Design based on Root Locus Lead Compensation

$$C(s) = K \frac{s + z}{s + p}, \quad z < p$$



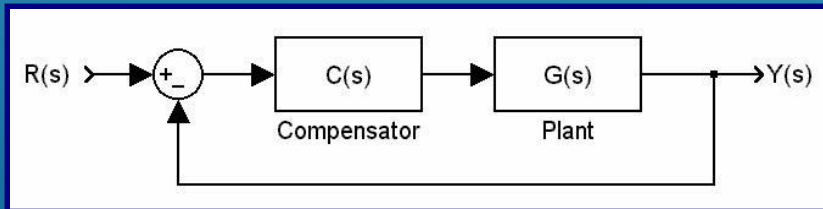
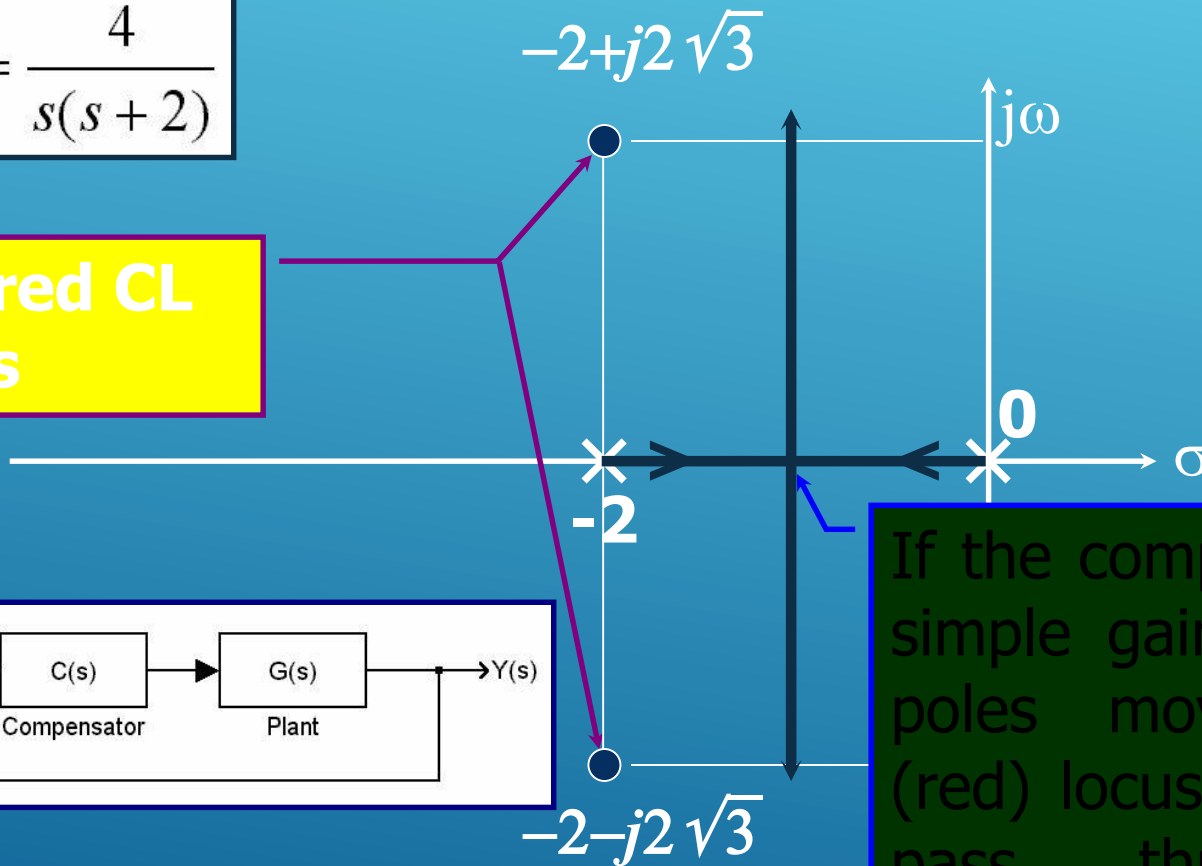
Calculate the angle deficiency (ϕ) at given locations, which are the desired CL pole locations, and then locate p and z to provide the $-\phi$ to satisfy angle condition. Then calculate K from the magnitude condition.

Design based on Root Locus

Lead Compensation - An Example

$$G(s) = \frac{4}{s(s+2)}$$

Desired CL poles

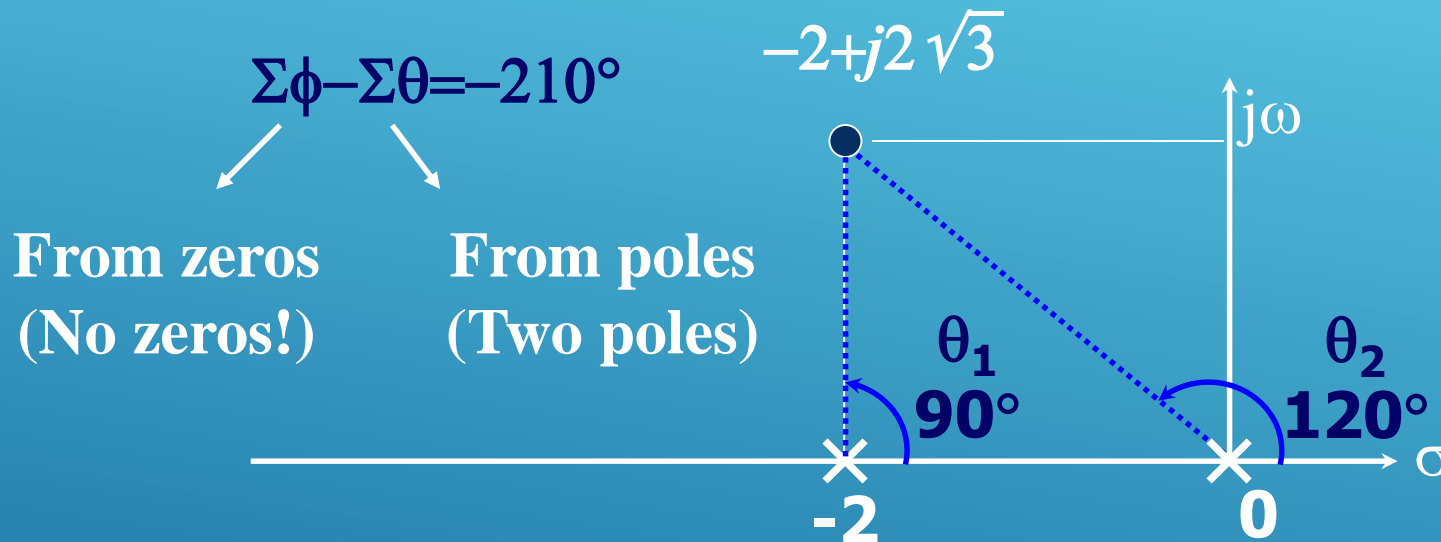


If the compensator is a simple gain, K , the CL poles move on this (red) locus, and do not pass through the desired locations...

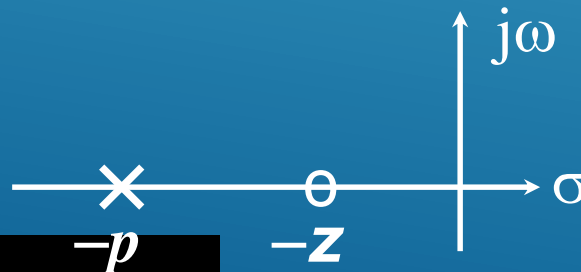
Design based on Root Locus

Lead Compensation - An Example

Check Angle Condition

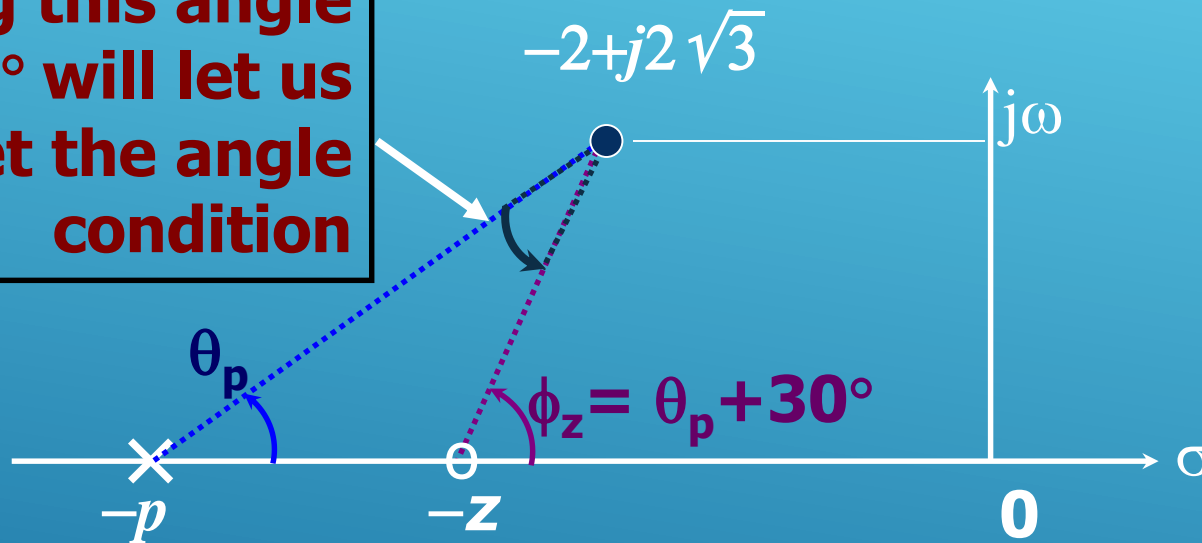


Compensator must provide 30° to satisfy the angle condition. Remember the p & z configuration of the lead compensator.



Design based on Root Locus Lead Compensation - An Example

Keeping this angle
at 30° will let us
meet the angle
condition

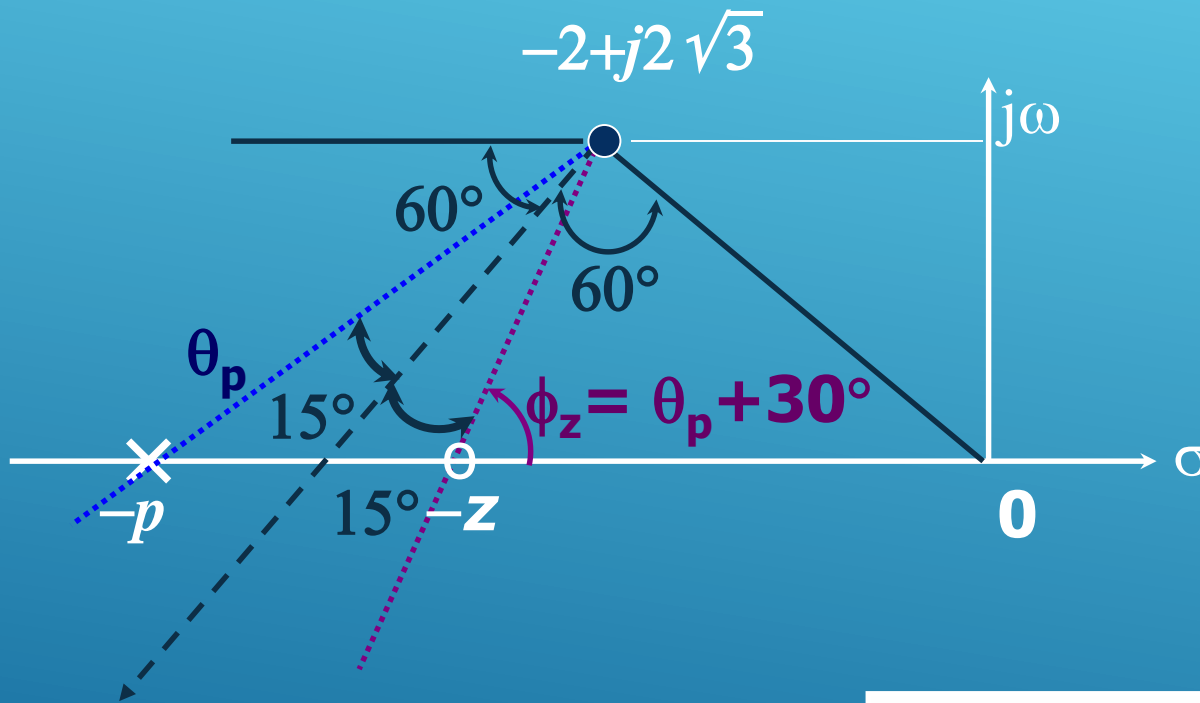


Clearly, there are lots of configurations providing 30° angle contribution? Which one should we choose?

Design based on Root Locus

Lead Compensation - An Example

To obtain the best K_v



Bisect line

$$p = 5.4$$

$$z = 2.9$$

$$C(s) = K \frac{s + 2.9}{s + 5.4}$$

Design based on Root Locus

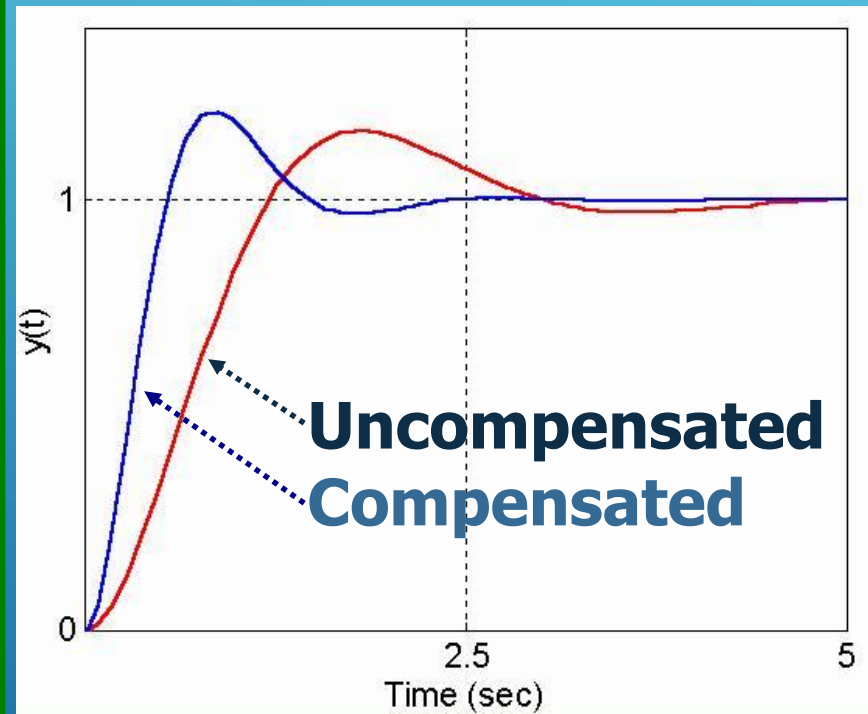
Lead Compensation - An Example

Determine K from the Magnitude Condition

$$|C(s)G(s)|_{s=-2 \pm j2\sqrt{3}} = 1$$

$$\left| K \frac{s + 2.9}{s + 5.4} \frac{4}{s(s + 2)} \right|_{s=-2 + j2\sqrt{3}} = 1$$

$$K = 4.7, \quad C(s) = 4.7 \frac{s + 2.9}{s + 5.4}$$



Design based on Root Locus

Lead Compensation - An Example

Static Velocity Error Constant

Input Type \ System Type	Step Input $r(t) = 1$	Ramp Input $r(t) = t$	Acceleration Input $r(t) = t^2/2$
Type 1	0	$e_{ss} = \frac{1}{K_v}$	∞

$$\begin{aligned}
 K_v &= \lim_{s \rightarrow 0} sC(s)G(s) \\
 &= \lim_{s \rightarrow 0} s \left(4.7 \frac{s + 2.9}{s + 5.4} \right) \left(\frac{4}{s(s + 2)} \right) \\
 &= 5.02 \text{ sec}^{-1}
 \end{aligned}$$

We obtained the maximum possible K_v , as the bisect line is chosen

Design based on Root Locus

Lead Compensation

More general case: **You are specified K_v**

$$K_v = \lim_{s \rightarrow 0} sC(s)G(s) = \lim_{s \rightarrow 0} s \left(K \frac{s+z}{s+p} \right) \left(\frac{4}{s(s+2)} \right) = \frac{2Kz}{p}$$

$$\phi_z - \theta_p = 30^\circ = \arctan\left(\frac{2\sqrt{3}}{z-2}\right) - \arctan\left(\frac{2\sqrt{3}}{p-2}\right)$$



$$\left| K \frac{s+z}{s+p} \frac{4}{s(s+2)} \right|_{s=-2+j2\sqrt{3}} = 1 \Rightarrow K = 2\sqrt{3} \sqrt{\frac{(p-2)^2 + 12}{(z-2)^2 + 12}}$$

Angle Condition

Magnitude Condition

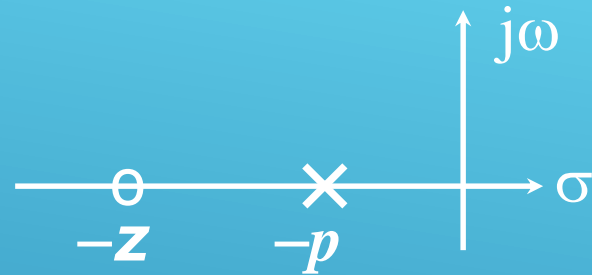
Solve the three equations for z , p and K

Design based on Root Locus Lead Compensation - Remarks

-  You have been given the CL poles explicitly in this example. In a more realistic problem, several specifications imply them. For example, the transient or steady state characteristics are described and you find out the required CL poles.
-  Before jumping into equations, roughly sketch the root loci and make sure that you are on the right way.

Design based on Root Locus Lag Compensation

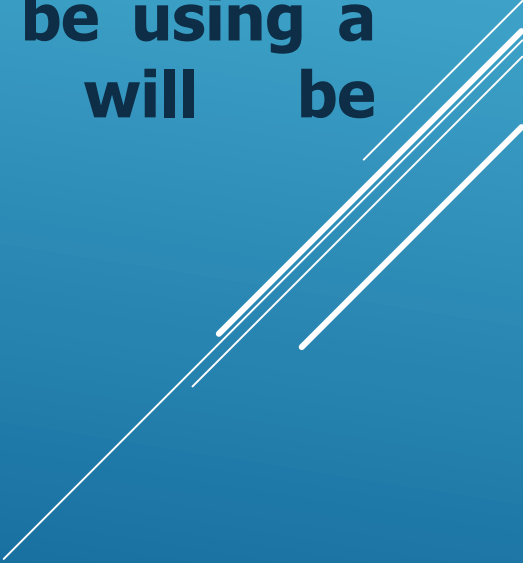
$$C(s) = K \frac{s + z}{s + p}, \quad z > p$$



If the system performs well during transient period but poor during steady state, use a lag compensator to improve the steady state characteristics. Lag compensator increases the loop gain without modifying the locations of the dominant CL poles significantly. This is true as long as you locate p and z close to each other, furthermore, both are located close to origin.

Design based on Root Locus Lag Compensation

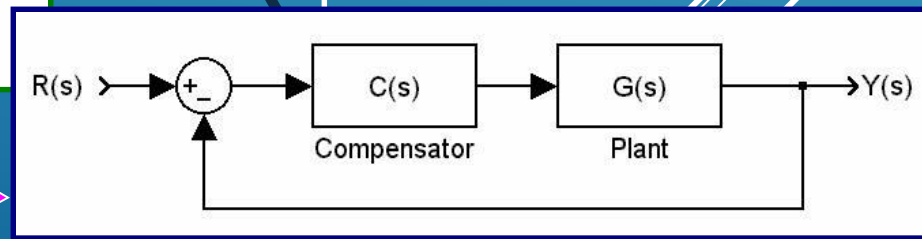
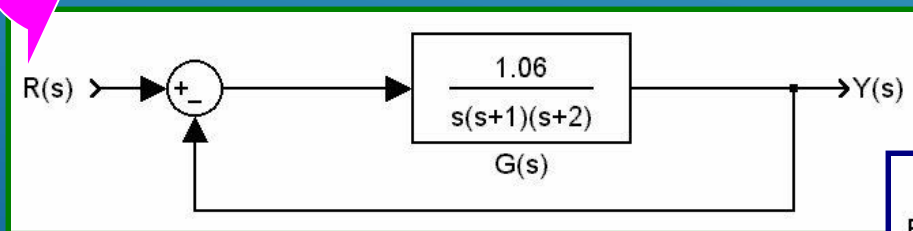
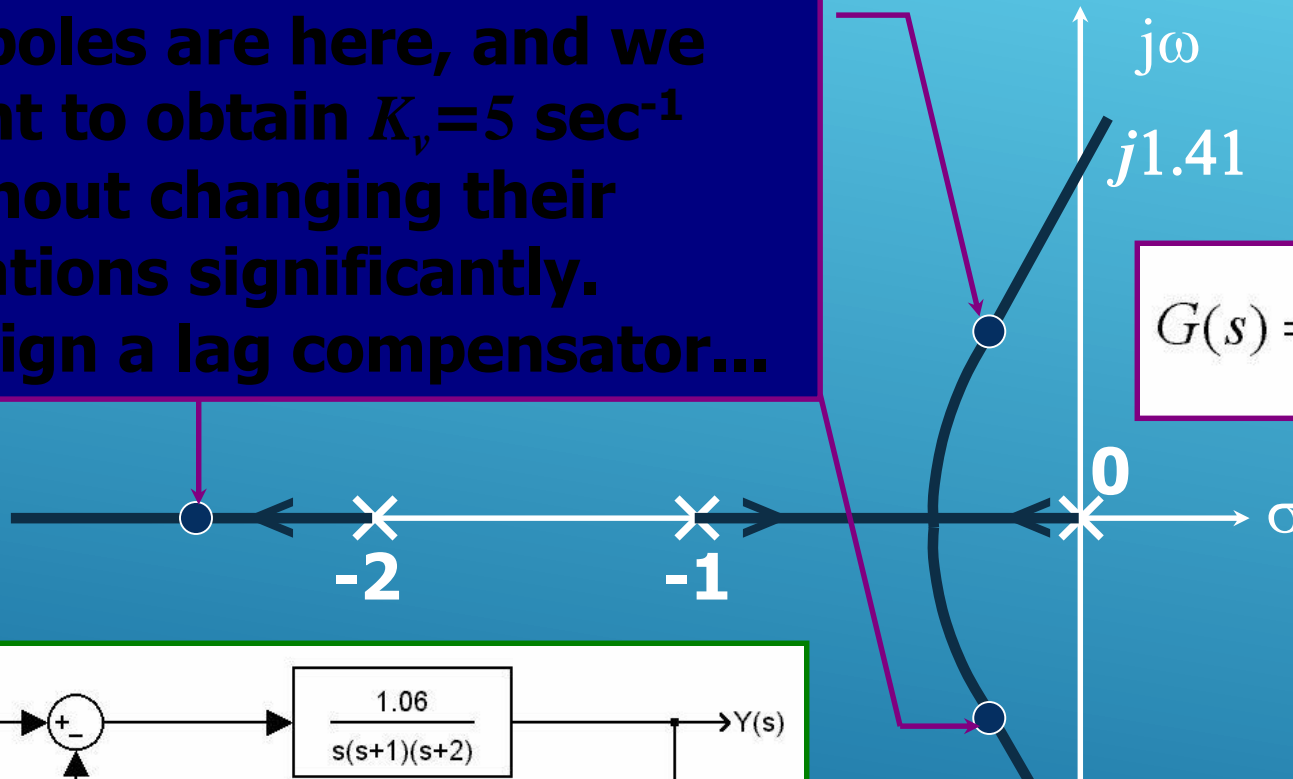
Typically, a desired static error constant is given. Since the angle contribution of the lag compensator is very small, the root loci does not change significantly. If this is not the case, i.e. if transient response is not satisfactory either, then you will be using a lag-lead compensator, which will be considered later...



Design based on Root Locus Lag Compensation - An Example

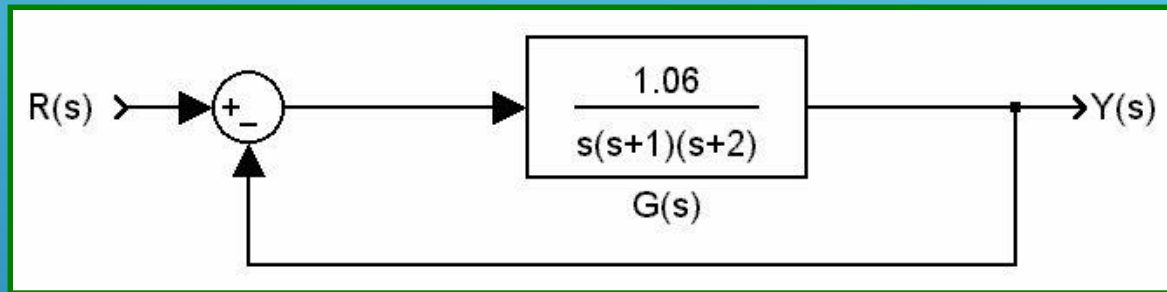
CL poles are here, and we want to obtain $K_v = 5 \text{ sec}^{-1}$ without changing their locations significantly. Design a lag compensator...

$$G(s) = \frac{1.06}{s(s+1)(s+2)}$$



Design based on Root Locus Lag Compensation - An Example

With this configuration,



**The dominant CL poles are at
 $s = -0.3307 \pm j0.5864$**



The damping ratio is $\zeta = 0.491$



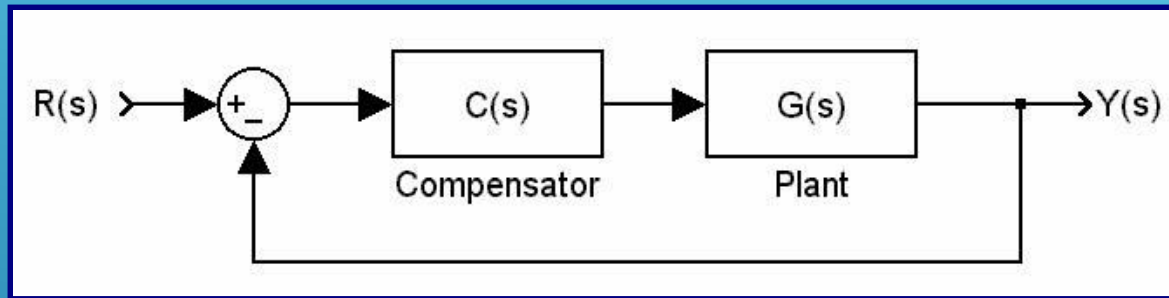
**The static velocity error constant is
 $K_v = 0.53 \text{ sec}^{-1}$**



We want $K_v = 5 \text{ sec}^{-1}$

Design based on Root Locus Lag Compensation - An Example

Adopt this configuration,



Locate z and p very close to origin

$$C(s) = K \frac{s + z}{s + p}, \quad z \gg p$$

Design based on Root Locus Lag Compensation - An Example

$$G(s) = \frac{1.06}{s(s+1)(s+2)} \quad K_v = \lim_{s \rightarrow 0} sG(s) = 0.53$$

$$C(s)G(s) = K \frac{s+z}{s+p} \frac{1.06}{s(s+1)(s+2)}$$

$$K_{v\text{NEW}} = \lim_{s \rightarrow 0} sC(s)G(s) = K \frac{z}{p} 0.53$$

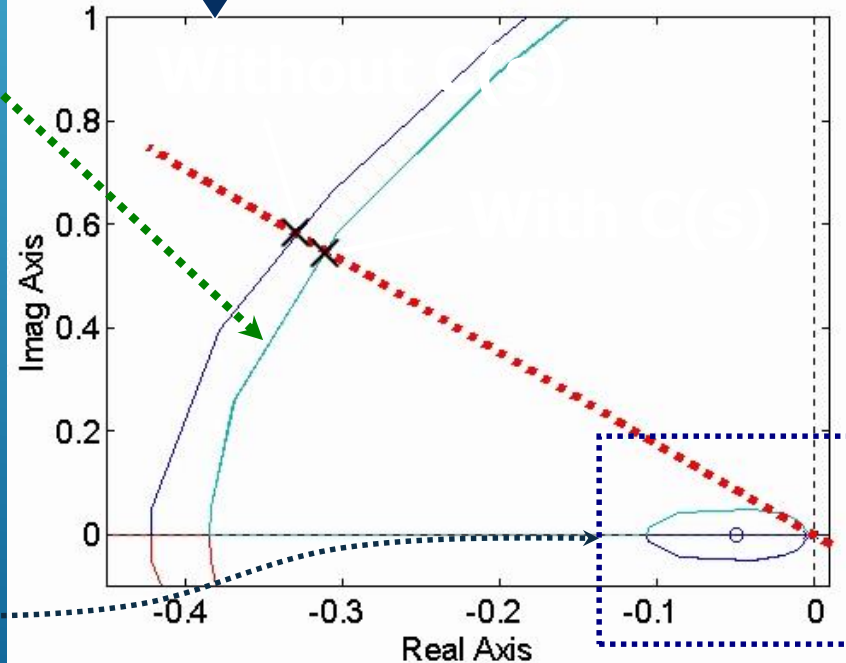
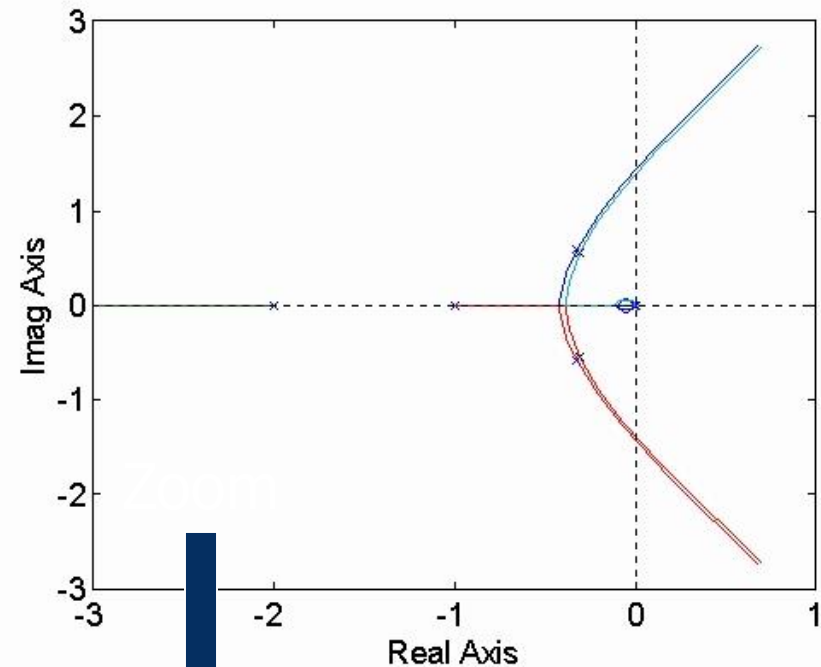
- $K_{v\text{NEW}}/K_v \approx 10$, so set $z=0.05$ and $p=0.005$
- Calculate angle contribution, which is $\approx 4^\circ$
- This will slightly change the root locus
- Tune K to keep ζ same ($\zeta=0.491$), $K=1.0235$

Design, R-Locus Lag Comp. Example

- What would happen if there were no K adjustment?

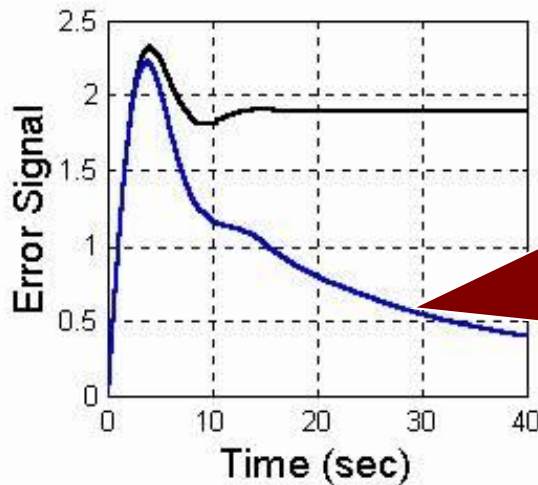
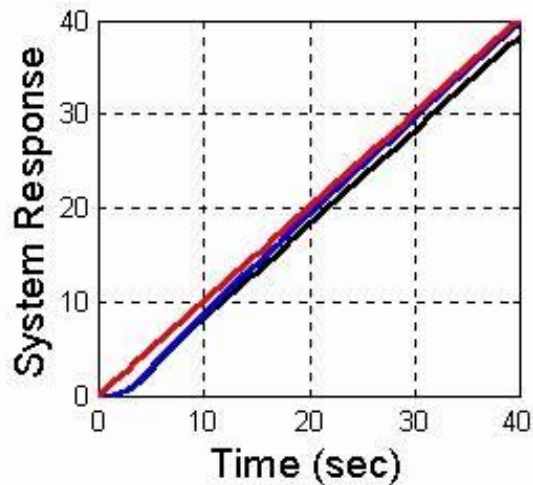
The answer is on the graph. Here you see two loci, which are almost identical. Nevertheless, you have to find the correct value of K...

Pay attention, the pole and the zero of $C(s)$ are here

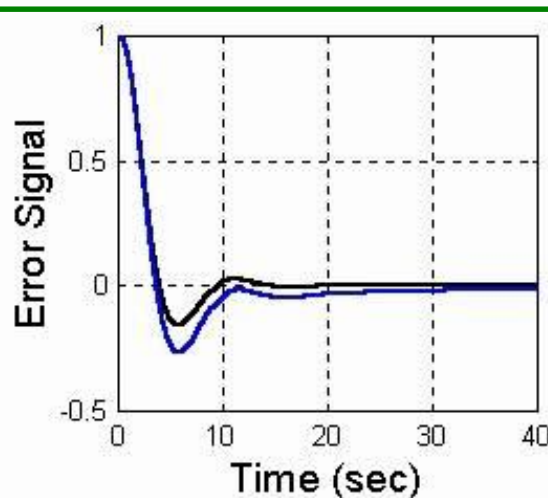
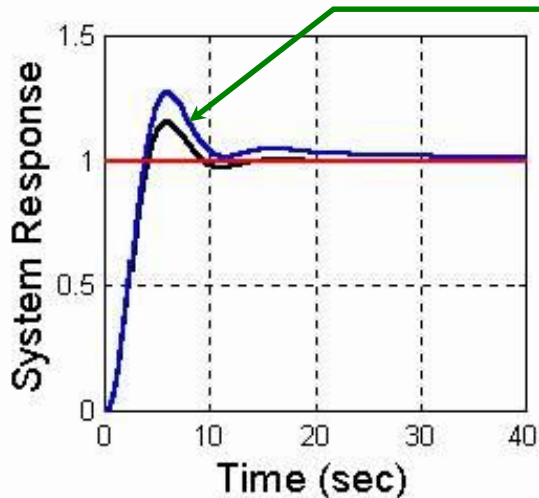


Design based on Root Locus Lag Compensation - An Example

Red: Command Signal, Blue: Compensated, Black: Uncompensated






This was what we aimed. Curve goes to 0.1



Speed of the response has decreased because of the lag compensation

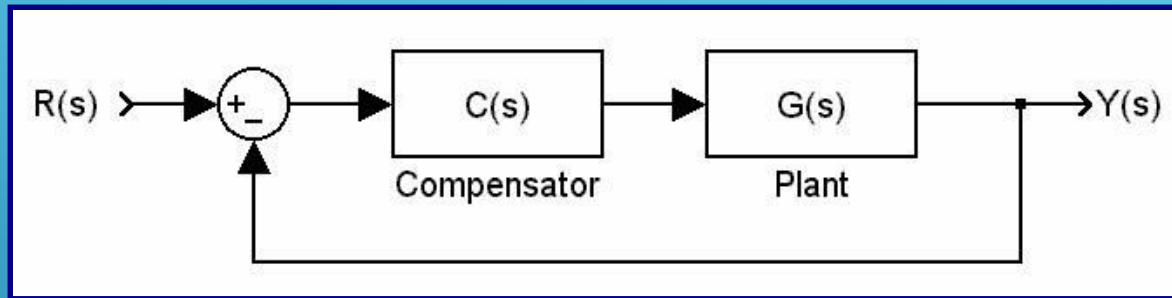
Design based on Root Locus Lag-Lead Compensation

-  Lead compensation speeds up the response and increases the stability of the system.
-  Lag compensation improves the steady state accuracy but reduces the speed of the response.
-  If the design specifications require both a fast response and better steady state characteristics, a Lag-Lead compensator is used.

Design based on Root Locus Lag-Lead Compensation

- Calculate the relevant variables (ω_n , ζ , ω_d etc)
- Firstly, design the Lead Compensator
 - Calculate the angle deficiency
 - Locate the zero of the compensator
 - Locate the pole such that the angle condition is met
- Secondly, design the Lag Compensator
 - Locate its pole close to zero
 - According to steady state response specs., locate the zero
 - Check the angle contribution of Lag Comp.
 - If necessary, retune the gain so that ζ is kept at its desired value.

Design based on Root Locus Lag-Lead Compensation - An Example



$$G(s) = \frac{4}{s(s + 0.5)}$$

Design Specifications

- Dominant CL poles are desired to have $\zeta = 0.5$
- Desired Undamped natural frequency is $\omega_n = 5$ rad/sec
- Desired Static velocity error constant is $K_v = 80$ sec⁻¹

Design based on Root Locus

Lag-Lead Compensation - An Example

Step 1: Calculate the relevant variables

If there is no compensator, you have

$$G(s) = \frac{4}{s(s + 0.5)}$$

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{4}{s^2 + 0.5s + 4}$$

Desired
 $\zeta = 0.5$

Desired
 $K_v = 80 \text{ sec}^{-1}$

$$\zeta = 0.125 \text{ and } \omega_n = 2 \text{ rad/sec}$$

$$K_v = \lim_{s \rightarrow 0} sG(s) = 8 \text{ sec}^{-1}$$

Desired
 $\omega_n = 5 \text{ rad/sec}$

Design based on Root Locus

Lag-Lead Compensation - An Example

Step 2: Design the Lead Compensator

$$-\zeta\omega_n = -0.5 * 5 = -2.5$$

$$j\omega_d = j\sqrt{1-\zeta^2}\omega_n = j4.3301$$

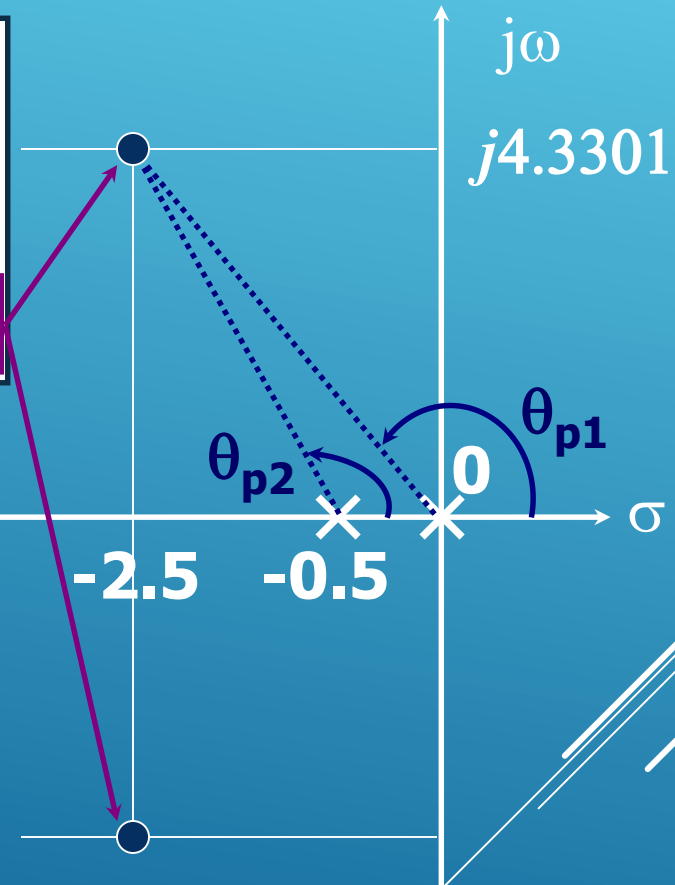
$$s_{1,2} = -\zeta\omega_n \pm j\omega_d = -2.5 \pm j4.3301$$

Now Calculate the angle deficiency

$$\Sigma\phi - \Sigma\theta = -234.8^\circ$$

Lead Controller will contribute

54.8° to make sure that $\Sigma\phi - \Sigma\theta = \pm(2k+1)180^\circ$



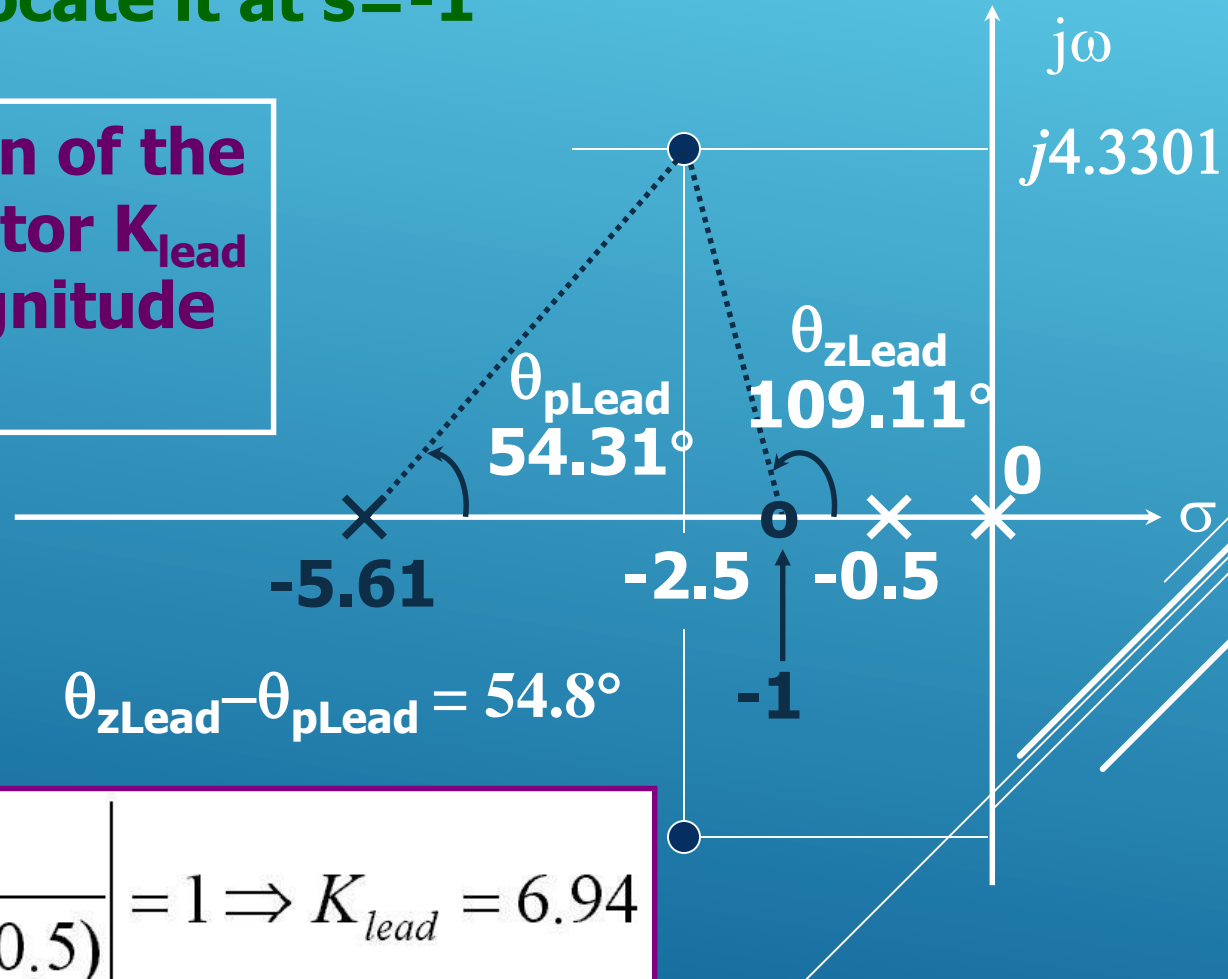
Design based on Root Locus

Lag-Lead Compensation - An Example

Step 2: Locate the zero of Lead Comp.

Let's locate it at $s=-1$

Now set the gain of the Lead Compensator K_{lead} i.e. refer to magnitude condition



$$K_{lead} \frac{s+1}{s+5.61} \frac{4}{s(s+0.5)} = 1 \Rightarrow K_{lead} = 6.94$$

Design based on Root Locus

Lag-Lead Compensation - An Example

Step 3: Design the Lag Compensator

$$C_{lead}(s)G(s) = \left(6.94 \frac{s+1}{s+5.61} \right) \left(\frac{4}{s(s+0.5)} \right)$$

$$K_{v(new)} = \lim_{s \rightarrow 0} s C_{lead}(s)G(s) = 9.9 \text{ sec}^{-1}$$

$$\frac{K_{v(new)}}{K_{v(desired)}} = \frac{9.9 \text{ sec}^{-1}}{80 \text{ sec}^{-1}} = 0.124$$

When $s=0$, the Lag compensator must increase the loop gain by $1/0.124 \approx 8.06$

Design based on Root Locus Lag-Lead Compensation - An Example

**Step 3: Locate the zero
of Lag Compensator
Let's locate it at $s=-0.1$**

$$C_{lag}(s) = K_{lag} \frac{s + 0.1}{s + 0.0124}$$

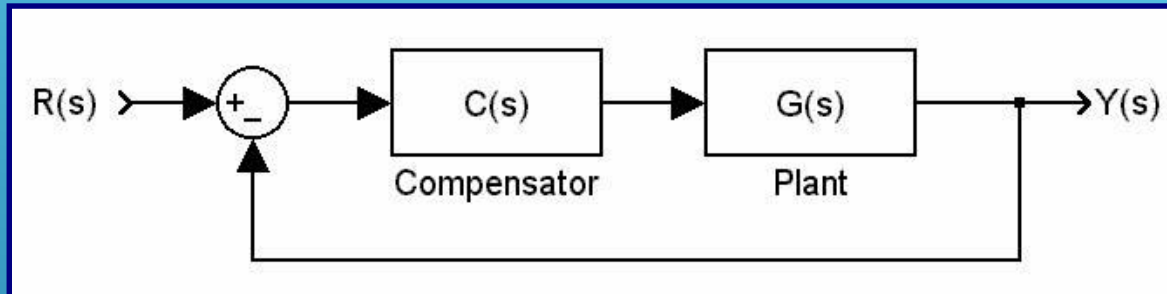
$$C_{lead}(s)C_{lag}(s)G(s) = \left(6.94 \frac{s + 1}{s + 5.61} \right) \left(K_{lag} \frac{s + 0.1}{s + 0.0124} \right) \left(\frac{4}{s(s + 0.5)} \right)$$

$$K_v = \lim_{s \rightarrow 0} s C_{lead}(s) C_{lag}(s) G(s) = K_{lag} 79.8114 \text{ sec}^{-1}$$

Angle Contribution is : 0.8791°

Angle contribution is acceptably small. However, this has slightly changed z . A very tiny tuning can be made if the design specifications are too stringent. For this example, there is no need to do so, keep $K_{lag} = 1$.

Design based on Root Locus Lag-Lead Compensation - An Example

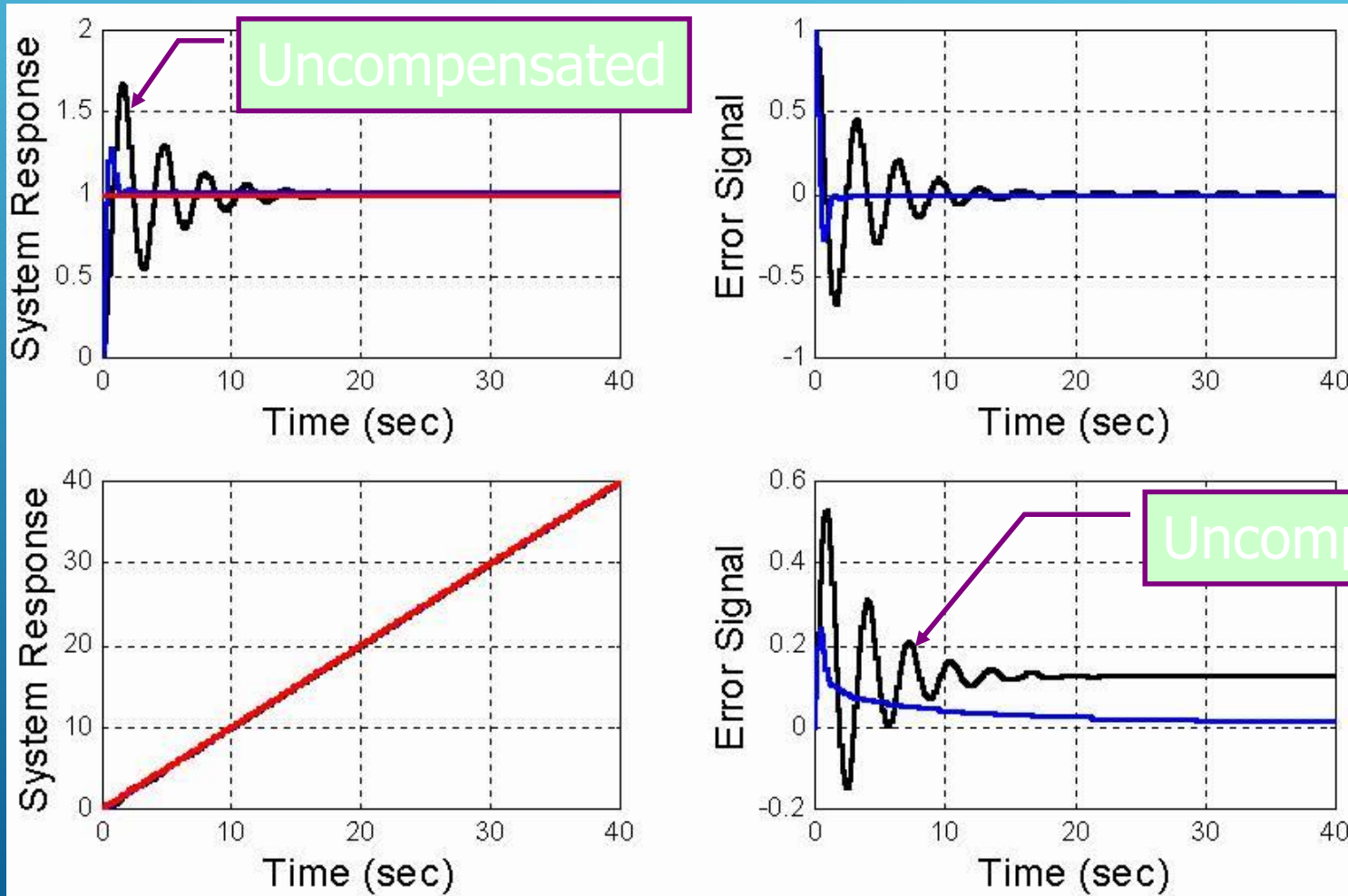


$$G(s) = \frac{4}{s(s + 0.5)}$$

$$C(s) = 6.94 \frac{(s + 1)(s + 0.1)}{(s + 5.61)(s + 0.0124)}$$

Now, test and see whether the design specifications are met or not...

Design based on Root Locus Lag-Lead Compensation - An Example Step and Ramp Responses



Design based on Root Locus Lag-Lead Compensation - An Example A Comparison

Simple K_{lag}
Good enough

$$K_{lag} = 1$$

$$K_v = 79.81139669944224 \text{ sec}^{-1}$$

$$\zeta = 0.49452458450471$$

$$\text{CL Poles: } s = -2.44946613086810 \pm j4.30511842727874 \\ s = -1.12268288809756 \text{ and } s = -0.10078485016624$$

K_v is exact

$$K_{lag} = 80/79.81139669944224 = 1.00236311239193$$

$$K_v = 80 \text{ sec}^{-1}$$

$$\zeta = 0.49388974530242$$

$$\text{CL Poles: } s = -2.44966485404744 \pm j4.31279190736033 \\ s = -1.12228732098688 \text{ and } s = -0.10078297091824$$

ζ is exact




$$K_{lag} = 0.97999709075950$$

$$K_v = 78.21493657490576 \text{ sec}^{-1}$$

$$\zeta = 0.500000000000001$$

$$\text{CL Poles: } s = -2.44773023820451 \pm j4.23959313579281 \\ s = -1.12613839738740 \text{ and } s = -0.10080112620358$$

Remarks on Root Locus and Design Based on Root Locus

-  **Manipulating the roots and the poles of the closed loop system may yield the desired solution, which can be sought by root locus method.**
-  **Stringent design specs. carry priority. Meeting them precisely may require computer based analysis and design.**
-  **It is useful to know the following Matlab functions: `rlocus(.,.)`, `rlocfind(.,.)` and `rltool`. The last one lets you play with the poles and zeros to see their effects on responses and several other control engineering design tools.**