Lecture 6: Spanning Set & Linear Independency

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Spanning Set & Linear Independency

Definition (Linear Combination)

Let $v_1, v_2, ..., v_k$ be vectors in $(V, \oplus, \odot)$ a vector space. A vector $v \in V$ is called a linear combination of $v_1, v_2, ..., v_k$ if

$$v = c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus \ldots \oplus c_k \odot v_k,$$

for the scalars $c_1, c_2, ..., c_k$.

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Definition (Spannig Set)

Let $S = \{v_1, v_2, ..., v_k\}$ be a set of vectors in a vector space V. Then the set of all vectors in V that are linear combinations of the vectors in S is denoted by $SpanS$, that is,

$$SpanS = \{c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus \ldots \oplus c_k \odot v_k \mid c_1, c_2, ..., c_k \in \mathbb{R}\}.$$

If every vector in V is a linear combination of the vectors in S, the set S is said to span V and denoted by $SpanS = V$.

Theorem

Let $S = \{v_1, v_2, ..., v_k\}$ be a set of vectors in a vector space V. Then $SpanS < V$.

Note that a vector space may have many spanning sets.

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To determine whether a vector v of V is in $Span S$, we investigate the consistency of the corresponding linear system.

Example

Let $V := \mathbb{R}^3$, consider the vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}.$$

Determine whether $Span\{v_1, v_2, v_3\} = V$.

Solution: Let $v := \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be an arbitrary vector in $\mathbb{R}^3$. If we find the scalars c_1, c_2, and c_2 such that $v = c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus c_3 \odot v_3$, then the set $\{v_1, v_2, v_3\}$ spans $\mathbb{R}^3$.

The corresponding linear system is

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= a \\ 2c_1 + c_2 - 3c_3 &= b \\ c_1 - c_2 + c_3 &= c. \end{aligned}$$

If we transform the augmented matrix to the reduced row echelon form, we obtain

$$\begin{aligned} c_1 &= (19a - 5b - 9c)/21 \\ c_2 &= (5a + 2b - 9c)/21 \\ c_3 &= (a - b + c)/7 \end{aligned}$$

which indicates the corresponding linear system is consistent. Thus, $Span\{v_1, v_2, v_3\} = V$.

Definition (Linear independency)

The vectors $v_1, v_2, ..., v_k$ in a vector space $(V, \oplus, \odot)$ are said to be *linearly independent* if there exist scalars $c_1, c_2, ..., c_k$, not all zero, such that

$$c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus \ldots \oplus c_k \odot v_k = 0.$$

The vectors $v_1, v_2, ..., v_k$ are called linealy independent if $c_1 = c_2 = \cdots = c_k = 0$ such that

$$c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus \ldots \oplus c_k \odot v_k = 0.$$

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To determine whether a set of vectors is linearly independent or linarly dependent, we investigate the nontrivial (nonzero) solution of the corresponding homogenous linear system. If the system has a nontrivial solution, then the vectors are linearly dependent. If the system has only trivial (zero) solution, then the vectors are linearly independent.

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Example

Determine whether the vectors $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$ are linearly independent.

Solution: The corresponding homogenous linear system is

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ 2c_1 + c_2 - 3c_3 &= 0 \\ c_1 - c_2 + c_3 &= 0. \end{aligned}$$

If we transform the augmented matrix to the reduced row echelon form, we obtain $c_1 = c_2 = c_3 = 0$ which indicates the linear system has only zero solution. Thus the vectors are linearly independent.

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Theorem

Let $S = \{v_1, v_2, ..., v_n\}$ *be a set of vectors in a vector space* $\mathbb{R}^n$*. Let* A *be the matrix whose columns are the elements of* S*. Then* S *is linearly independent if and only if* $\det(A) \neq 0$.

Example

From the former example, since

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix} \approx \cdots \approx I_3,$$

then $\det(A) \neq 0$ which indicates the vectors are linearly independent.

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Theorem

1. *Let S_1 and S_2 be finite subset of a vector space V and $S_1 \subset S_2$. Then*

$$(i)\ \ S_1 \text{ is linearly dependent} \Rightarrow S_2 \text{ is linearly dependent}$$

$$(ii)\ \ S_2 \text{ is linearly independent} \Rightarrow S_1 \text{ is linearly independent.}$$

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2. *If S is any set of vectors that contains 0, then S is linearly dependent.*

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3. *If* S *consists of a single nonzero vector, then* S *is linearly independent.*

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4. *If* $S = \{v_1, v_2, ..., v_k\}$ *is linearly independent, then the vectors* $v_1, v_2, ..., v_k$ *must be distinct and nonzero.*

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4. *If* $S = \{v_1, v_2, ..., v_k\}$ *is linearly independent, then the vectors* $v_1, v_2, ..., v_k$ *must be distinct and nonzero.*
5. *Let* $S = \{v_1, v_2, ..., v_k\}$ *spans a vector space* V *and* v_j *is a linear combination of the preceding vectors in* S. *Then the set* $S - \{v_j\}$ *also spans* V.