Lecture 7: Basis and Dimension

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Basis and Dimension

Definition

Basis: Let $v_1, v_2, ..., v_k$ be vectors in a vector space $(V, \oplus, \odot)$. The vectors $v_1, v_2, ..., v_k$ are said to form a basis for V if
(i) $Span\{v_1, v_2, ..., v_k\} = V$
(ii) $\{v_1, v_2, ..., v_k\}$ is linerly independent.
Dimension: The number of vectors in a basis for the vector space V is called as a dimension of V $(dimV)$. The dimension of the zero vector space $\{0\}$ is defined as zero.

A vector space can have many different basis but the dimension of the vector space is always same.

Example

$S := \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis (standard basis) for the vector space $\left(\mathbb{R}^3, \oplus, \odot\right)$ and $dim\mathbb{R}^3 = 3$.
Generally, the standard basis for the vector space $\mathbb{R}^n$ is defined by $S = \{e_1, e_2, ..., e_n\}$, where e_j is an $n \times 1$ matrix whose j-th row is 1 and zero elsewhere.

Theorem

If $S = \{v_1, v_2, ..., v_n\}$ is a basis for a vector space V, then every vector in V can be written uniquely as a linear combination of the vectors in S.

Theorem

Let $V = \mathbb{R}^m$, $S = \{v_1, v_2, ..., v_n\}$, $(n \geq m)$ *be a set of nonzero vectors in* V *and Span*$S = W$. *Then some subset of* S *is a basis for* W. *The procedure for finding this basis is in the following:*

1. *Form equation* $c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus \ldots \oplus c_n \odot v_n = 0$

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1. *Form equation* $c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus \ldots \oplus c_n \odot v_n = 0$
2. *Construct the augmented matrix associated with the corresponding homogenous linear system and transform it to the reduced row echelon form.*

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1. *Form equation $c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus \ldots \oplus c_n \odot v_n = 0$*
2. *Construct the augmented matrix associated with the corresponding homogenous linear system and transform it to the reduced row echelon form.*
3. *The vectors corresponding to the columns containing the leading 1's form a basis for W.*

Example

$V := \mathbb{R}^3$,
$S :=$
$$\left\{ v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} -1 \\ 7 \\ 1 \end{bmatrix}, v_5 = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} \right.$$
It is easy to show that $SpanS = V$. Find a subset of S that is a basis for $\mathbb{R}^3$.

Solution:

$$c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus c_3 \odot v_3 \oplus c_4 \odot v_4 \oplus c_5 \odot v_5 = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & -1 & 5 \\ 2 & 1 & -3 & 7 & -2 \\ 1 & -1 & 1 & 1 & 0 \end{bmatrix} \approx \cdots \approx \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}.$$

Then, the leading 1's appears in columns 1,2,3, so $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.

Basis and Dimension

Theorem

1. *If* $S = \{v_1, v_2, ..., v_n\}$ *is a basis for* V *and* $W = \{w_1, w_2, ..., w_r\}$ *is a linearly independent set of vectors in* $V \Rightarrow r \leq n$.

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2. *If* $S = \{v_1, v_2, ..., v_n\}$ *is a basis for* V *and* $U = \{u_1, u_2, ..., u_m\}$ *spans* $V \Rightarrow m \geq n$.

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3. *Let* V *be an n-dimensional vector space. If* $S = \{v_1, v_2, ..., v_n\}$ *is linearly independent set of* $V \Rightarrow S$ *is a basis for* V.

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3. *Let* V *be an n-dimensional vector space. If* $S = \{v_1, v_2, ..., v_n\}$ *is linearly independent set of* $V \Rightarrow S$ *is a basis for* V.
4. *Let* V *be an n-dimensional vector space. If* $S = \{v_1, v_2, ..., v_n\}$ *spans* $V \Rightarrow S$ *is a basis for* V.

Example

$V := \mathbb{R}^3$,

$$S := \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} \right\}.$$

$Span S = V$. From the above theorem S is not a basis for $\mathbb{R}^3$.