Solution of partial differential equations (The laplace transform method)

Example

The death of the fish in a lake where is at the top of a high mountain is due to freezing of the surface resulting reduced oxygen levels. At the end of the winter following the melting of ice, concentration of oxygen at the lake was found to be $C_{A0}=4x10^{-5}$ kmol/m^3. The **stagnant lake** became enriched by O_2 at spring because it had been contact with air.

a. Calculate the O_2 concentration (C_A) 5 cm deep for the 1st day. Considering the stagnant lake is at 2500 m altitude and 5 ^{0}C temperature.

b. Find the penetration depth after 30 days.

Note: O_2 concentration at the interface is assumed to be at equilibrium with air in contact and can be taken

$C_A^* = 4x10^{-4}$ kmol/m^3. Diffusion coefficient is $2x10^{-9}$ kmol/m^3. Oxygen is only transferred **one dimensional (z direction).**

Use $y = \dfrac{C_A - C_{A0}}{C_A^* - C_{A0}}$ to solve equation with **Laplace.**

Solution

$$N_A A\big|_z - N_A A\big|_{z+\Delta z} = \frac{\partial}{\partial t}(C_A.V); \quad V = A.\Delta z$$

divide by $A.\Delta z$ and lim $\Delta z \to 0$,

$$-\frac{\partial(N_A)}{\partial z} = \frac{\partial C_A}{\partial t}$$

$N_A = -D_{AB}\dfrac{\partial C_A}{\partial z} + v_z C_A$ Because of stagnant lake,convective transport is negligible with respect to diffusion.

$$N_A = -D_{AB}\frac{\partial C_A}{\partial z}$$

$$D_{AB}\frac{\partial^2 C_A}{\partial z^2} = \frac{\partial C_A}{\partial t}$$

$$y = \frac{C_A - C_{A0}}{C_A^* - C_{A0}}$$

$$D_{AB}\frac{\partial^2 y}{\partial z^2} = \frac{\partial y}{\partial t}$$

IC:	t=0	$C_A=C_{A0}$	y=0	$\bar{y} = 0$
BC1:	z=0	$C_A=C_A$*	y=1	$\bar{y} = 1/s$
BC2:	$z \rightarrow \infty$	$C_A=C_{A0}$	y=0	$\bar{y} = 0$

Laplace Transformation

$$L\left\{\frac{\partial y}{\partial z}\right\} = \int_0^\infty \frac{\partial y}{\partial z}.e^{-st}.dt = \frac{\partial}{\partial z}\int_0^\infty y.e^{-st}.dt = \frac{\partial \bar{y}}{\partial z}$$

$$L\left\{\frac{\partial y}{\partial t}\right\} = \int_0^\infty \frac{\partial y}{\partial t}.e^{-st}.dt \qquad e^{-st} = u \qquad \frac{\partial y}{\partial t}dt = dv$$

$$-s.e^{-st}.dt = du \qquad y = v$$

$$L\left\{\frac{\partial y}{\partial t}\right\} = e^{-st}y\Big|_0^\infty - \int_0^\infty y.-s.e^{-st}.dt = e^{-st}y\Big|_0^\infty + s\int_0^\infty y.e^{-st}.dt = e^{-\infty}y(\infty) - e^0 y(0) + s\bar{y}$$

$$L\left\{\frac{\partial y}{\partial t}\right\} = 0.y(\infty) - 1.0 + s\bar{y} = s\bar{y}$$

$$\frac{d^2\bar{y}}{dz^2} - \frac{s}{D_{AB}}\bar{y} = 0$$

$$\bar{y} = c_1 e^{-z\sqrt{s/D_{AB}}} + c_2 e^{z\sqrt{s/D_{AB}}}$$

Apply BC1 and BC2

General solution: $\bar{y} = \frac{1}{s}e^{-z\sqrt{s/D_{AB}}}$

Inverse laplace:

$$y = \frac{C_A - C_{A0}}{C_A^* - C_{A0}} = \operatorname{erfc}\left\{\frac{z}{2\sqrt{D_{AB}t}}\right\}$$

a. 1 day = 86400 s, z = $5*10^{-2}$ m

$$\frac{C_A - 4*10^{-5}}{4*10^{-4} - 4*10^{-5}} = 1 - \mathrm{erf}\left\{\frac{0.05}{2\sqrt{2*10^{-9}*86400}}\right\}$$

$C_A = 4.37*10^{-5}$ kmol/m^3

b. $C_A = C_{Ao}$ is penetration depth → $C_{Ao} = 4*10^{-5}$ kmol/m^3

$$0 = 1 - \mathrm{erf}\left\{\frac{z}{2\sqrt{2*10^{-9}*30*86400}}\right\}$$

$$\mathrm{erf}\{\infty\} = 1$$

$$\mathrm{erf}\left\{\frac{z}{0.144}\right\} = 1 \qquad \mathrm{erf}\{2\} \cong 1$$

z = 0.288 m