Lecture 2 :

The electromagnetic waves : Wave equations for the electric and magnetic fields can be derived starting from the Maxwell' equations. In vacuum and free space (namely no charge and current sources) we write them as :

$$\text{(i)} \quad \nabla \cdot \mathbf{E} = 0, \quad \text{(iii)} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\text{(ii)} \quad \nabla \cdot \mathbf{B} = 0, \quad \text{(iv)} \quad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

Taking the curl of the third equation, and using the vector identity for the product rule for del operators and also inserting the first and fourth equations one can easily arrive at the wave equation for electric field. Following similar steps starting with Ampere's + Maxwell law the wave equation for the magnetic field can be obtained. These two wave equations are :

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \quad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

Here $v = \sqrt{\frac{1}{\varepsilon_0 \mu_0}}$ = 299 792 458 m/s (today it is taken to be an exact value and is considered as one of the physical constants of nature) being speed of light and thus is a prediction of the Maxwell equations.

Plane waves propagating in the direction of the propagation vector **k,** whose magnitude is the wave number k, can be expressed in complex notation as follows. Physical fields measured at the laboratories are their real parts :

$$\tilde{\mathbf{E}}(\mathbf{r}, t) = \tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \, \hat{\mathbf{n}},$$

$$\tilde{\mathbf{B}}(\mathbf{r}, t) = \frac{1}{c} \tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} (\hat{\mathbf{k}} \times \hat{\mathbf{n}}) = \frac{1}{c} \hat{\mathbf{k}} \times \tilde{\mathbf{E}}.$$

Using the Maxwell equations (Coulomb's law in free space for example) one can immediately show that the wave is transverse, namely $\hat{n} \cdot \hat{k} = 0$. As can be seen from the second equation above the amplitued of the magnetic wave is B = E/c (in SI unit system).

Since energy is sored in the electric and magnetic fields one can associate an energy flow rate with electromagnetic waves.

$u = \frac{1}{2}(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2)$ is the energy density, and the Poynting vector

$\boldsymbol{S} = \frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B}$ is the energy flux density .

For a monochromatic plane wave which propagates in the positive z-direction one gets :

$$\boldsymbol{S} = c\,\varepsilon_0 E_0^2 \cos^2(kz - \omega t + \phi)\,\hat{z} = c\,u\,\hat{z}$$

Electromagnetic waves carry also momentum; momentum density stored in the fields is

$$\boldsymbol{p} = \frac{1}{C^2}\, \boldsymbol{S}$$

Again for a monochromatic wave it takes the following form which is also known from the theory of special relativity :

$$\boldsymbol{p} = \frac{1}{c^2} c\, \varepsilon_0\, E_0^2 \cos^2(kz - \omega t + \phi)\,\hat{z} = \frac{1}{c} u\,\hat{z}$$

In practice usually we are interested in time average quantities. Since the time average of the cosine squared term introduces a factor of ½ we obtain the following relations :

$$\langle u \rangle = \frac{1}{2}\, \varepsilon_0\, E_0^2$$

$$\langle \boldsymbol{S} \rangle = \frac{1}{2}\, c\varepsilon_0\, E_0^2\, \hat{z}$$

$$\langle \boldsymbol{p} \rangle = \frac{1}{2c}\, \varepsilon_0\, E_0^2\, \hat{z}$$

One define the intensity *I* of the electromagnetic wave as the average power transported $\langle \boldsymbol{S} \rangle$:

$$I = \langle \boldsymbol{S} \rangle = \frac{1}{2}\, c\varepsilon_0\, E_0^2\, \hat{z}$$

When the electromagnetic wave falls on a material and is absorbed perfectly then it exerts a pressure P :

$$\mathrm{P} = \frac{force}{area} = \frac{1}{A}\,\frac{\Delta p}{\Delta t} = \frac{\langle \boldsymbol{p} \rangle A c \Delta t}{A \Delta t} = \frac{1}{2}\varepsilon_0\, E_0^2 = \frac{\langle S \rangle}{c} = \frac{I}{c}$$