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# Linear Algebra Review

## Lecture 3

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# Outline

# Linear Algebra Review

## Numbers

- Natural numbers: $\mathbb{N}$=$\{1, 2, 3, ..., \}$
- Integers numbers: $\mathbb{Z} = \{, ..., -1, 0, 1, ..., \}$
- Rational numbers are composed of ratios of integers: $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$
- Irrational numbers are numbers like $\pi$, $\sqrt{2}$, Euler's number, golden ratio, etc.
- Real numbers contains $\mathbb{Q}$ and irrational numbers but not imaginary numbers.
- let $i$ be $\sqrt{-1}$, complex numbers are: $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$

# Linear Algebra Review

## Vector Space

*Definition*: A *vector space* is a set $\mathcal{V}$ that is closed under addition and scalar multiplication.

*Example*: Let $p : \mathbb{R} \to \mathbb{R}$ and $p \in \mathbb{R}[x]$:

$$p(x) = \sum_k a_k x^k, \forall a_k \in \mathbb{R},$$

For any two polynomial $p_i, p_j \in \mathbb{R}[x]$:

$$c_1 p_i(x) + c_2 p_j(x) \in p(x), \forall c_1, c_2 \in \mathbb{R}$$

*Question*: Is $\mathbb{R}^n$ a vector space?

(Remember the definition of a vector space: A vector space is a set $\mathcal{V}$ that is closed under addition and scalar multiplication.)

For example, let $v_1 = (x_1, y_1) \in \mathbb{R}^2$ and $v_2 = (x_2, y_2) \in \mathbb{R}^2$:

$$\begin{aligned} &\forall c_1, c_2 \in \mathbb{R} : \\ &\quad c_1 v_1 + c_2 v_2 = (c_1 x_1, c_1 y_1) + (c_2 x_2, c_2 y_2) \\ &\qquad\qquad\quad\;\; = (c_1 x_1 + c_2 x_2, c_1 y_1 + c_2 y_2) \end{aligned}$$

since $(c_1 x_1 + c_2 x_2, c_1 y_1 + c_2 y_2) \in \mathbb{R}^2$, we can say it is closed when $n = 2$. If we check for any $n$:

$$\begin{aligned} v_1 &= (x_1, ..., x_n) \in \mathbb{R}^n \\ v_2 &= (y_1, ..., y_n) \in \mathbb{R}^n \\ c_1 v_1 + c_2 v_2 &= (c_1 x_1 + c_2 y_1, ..., c_1 x_n + c_2 y_n) \end{aligned}$$

Since, $(c_1 x_1 + c_2 y_1, ..., c_1 x_n + c_2 y_n) \in \mathbb{R}^n$, we conclude that $\mathbb{R}^n$ is a vector space.

# Linear Algebra Review

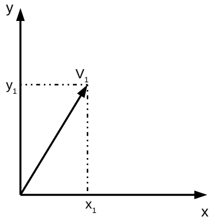
## Vectors

*Definition*: A vector is an element in a vector space that has a direction and a magnitude.

Throughout the course, vectors will be considered to be in column form:

$$\vec{v} = (x_1, x_2, ..., x_n) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Let $\vec{v_1} = (x_1, y_1)$ be a vector in $\mathbb{R}^2$:



*Dot Product*: Let $\vec{x} = (x_1, x_2, ..., x_n)$ and $\vec{y} = (y_1, y_2, ..., y_n)$ be two vectors in $\mathbb{R}^n$:

$$\vec{x} \cdot \vec{y} = \sum_{k=1}^{n} x_k y_k$$
$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos(\theta) \quad (1)$$

where, $\theta$ is the angle between two vectors and $\|\vec{x}\|$ is the norm (length) of the vector $\vec{x}$.

*Cauchy-Schwarz Inequality*:

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\|\|\vec{y}\| \tag{2}$$

The inequality can be understood and derived using Eq. 1 and remembering $\max(\cos(\theta)) = 1$.

Since $\cos(90^\circ) = 0$, the result of the dot product of two perpendicular vectors is 0.

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## Matrices

<u>*Definition*:</u> A matrix is an arrangement of numbers in a rectangular structure. An $m \times n$ matrix $A$ consists of $m$ rows and $n$ columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{3}$$

This matrix can also be seen as a combination of $n$ column vectors.

$$A = [\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}] : \vec{v_1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m,1} \end{bmatrix}, \vec{v_2} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m,2}, \end{bmatrix}, \ldots, \vec{v_n} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

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## Matrices

*Definition*: Identity matrix is an $n \times n$ matrix where all its diagonal elements are ones and the remaining elements are zeros.

$$\mathcal{I}_{n\times n} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, A_{m\times n}\mathcal{I}_{n\times n} = \mathcal{I}_{m\times m}A_{m\times n} = A_{m\times n}$$

*Matrix Multiplication*:

$$A_{m\times n}\vec{c} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} c_1 a_{11} + \cdots + c_n a_{1n} \\ c_1 a_{21} + \cdots + c_n a_{2n} \\ \vdots \\ c_1 a_{m1} + \cdots + c_n a_{mn} \end{bmatrix}$$

*Matrix Transpose*: Transpose of matrix is equivalent to the original matrix flipped around its diagonal. Hence, via taking the transpose, each column of a matrix becomes the corresponding row of that matrix. The transpose of $A$ in (3):

$$A^\top = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Some properties of matrix transpose:

$$(A^\top)^\top = A, (A+B)^\top = A^\top + B^\top, (AB)^\top = B^\top A^\top$$

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## Types of Matrices

*Nonsingular Matrix*: A square (regular) matrix is nonsingular (its inverse exists) if and only if its deteminant is not zero.

*Upper Triangular Matrix*:

$$U = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{mn} \end{bmatrix}$$

*Lower Triangular Matrix*:

$$L = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Determinant of $2 \times 2$ matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, det|A| = a_{11}a_{22} - a_{21}a_{12} \tag{4}$$

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix}, \qquad |det(A)| = 2 \times 8 - 1 \times 6 = 10 \tag{5}$$