System of Equations
Lecture 4

Dr. Görkem Saygılı

Department of Biomedical Engineering
Ankara University

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System of Equations

Linear Systems

Let's consider the following equations:

$$
\begin{aligned}
2x_1 - 4x_2 + 2x_3 &= 0 \\
3x_1 - 5x_2 + 4x_3 &= 5 \\
x_1 + 2x_2 + 2x_3 &= 11
\end{aligned}
$$

How do we solve this equation? How can we find the values for x_1, x_2 and x_3 such that the equations are satisfied?

System of Equations

Linear Systems

Such a system can be represented as:

$$A = \begin{bmatrix} 2 & -4 & 2 \\ 3 & -5 & 4 \\ 1 & 2 & 2 \end{bmatrix} \quad (1)$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \vec{y} = \begin{bmatrix} 0 \\ 5 \\ 11 \end{bmatrix} \quad (2)$$

$$A\vec{x} = \vec{y} \quad (3)$$

The unknowns in such a system can be found using a variety of different methods.

System of Equations

Gaussian Elimination

Gaussian elimination is one of the fundamental algorithms to solve linear system of equations. Gaussian elimination consists of two parts. The first part is called forward substitution. The aim is to obtain an upper triangular matrix by applying permutations, row scaling and elimination starting from the pivot element.

$$\left[\begin{array}{rrr|r} 2 & -4 & 2 & 0 \\ 3 & -5 & 4 & 5 \\ 1 & 2 & 2 & 11 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 3 & -5 & 4 & 5 \\ 1 & 2 & 2 & 11 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 5 \\ 0 & 4 & 1 & 11 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & -3 & -9 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 3 & 10 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

System of Equations

Gaussian Elimination

After upper triangular matrix is obtained, the second step of Gaussian elimination is started. This step is called Back substitution. Starting from the last element.

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

The back substitution is done until indentity matrix is obtained on the left side.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Forward substitution of Gaussian elimination aims to obtain an upper triangular matrix which eases to find the solution substantially. Similarly, LU Decomposition uses lower triangular and upper triangular matrices to reach final solution.

$$A\vec{x} = \vec{b}$$

To solve this equation, LU Decomposion uses lower and upper triangular matrices as follows:

$$A = LU \tag{4}$$

where, L and U are lower and upper triangular matrices, respectively.

L and U are as follows:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

In the next step, the elements are found by using Eq. 4:

$$A = LU$$

System of Equations

LU Decomposition

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21} & l_{12}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}l_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Immediately, we find the values of u_{11}, u_{12}, u_{13} and l_{21} and we continue until we find the elements of both matrices.

After revealing L and U, we have:

$$LU\vec{x} = \vec{b}$$

Let's introduce another vector $\vec{y}$ such that:

$$U\vec{x} = \vec{y} \tag{5}$$

and therefore:

$$L\vec{y} = \vec{b} \tag{6}$$

Last step of LU Decomposion involves solving first Eqn. 6 and then solving Eqn. 5 similar to Gaussian elimination. Remember than solving equations with upper and lower triangular matrices is easier.

Example: Let L and $\vec{b}$ be:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$$

$$\begin{aligned} y_1 &= 1 \\ 2y_1 + y_2 = 4 \rightarrow y_2 &= 2 \\ 3y_1 + y_2 + y_3 = 8 \rightarrow y_3 &= 3 \end{aligned}$$

System of Equations

LU Decomposition

Example (con't): U and $\vec{y}$ (from the previous result) are:

$$U = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 5 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 5 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{aligned} x_3 &= 3 \\ 5x_2 - x_3 = 2 \rightarrow x_2 &= 1 \\ x_1 - x_3 = 1 \rightarrow x_1 &= 4 \end{aligned}$$

System of Equations
Cramer's Rule

For a linear system such as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The solution can be found using the determinant of A:

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|A|}, x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{|A|}, x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{|A|}$$