Interpolation

Lecture 5

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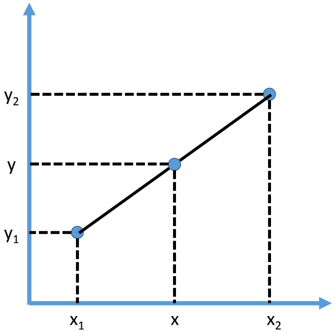
Interpolation

Given two data points, x_1 and x_2 and their values $f(x_1)$, $f(x_2)$, the goal of interpolation is to estimate the value of $f(x)$ for any $x \in (x_1, x_2)$.

Extrapolation

In contrast to interpolation, the aim of extrapolation is to estimate the value of the function $f(x)$ outside the interval $[x_1, x_2]$ ($x < x_1$ and $x > x_2$).

Line Fitting



Perhaps the simplest model to use for interpolation is fitting a straight line between the points and estimate the value of $f(x)$ using the fitted line.

To estimate the value of $f(x)$:

$$\frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = \frac{f(x) - f(x_0)}{(x - x_0)}$$

$$f(x) = \frac{(f(x_1) - f(x_0))(x - x_0)}{(x_1 - x_0)} + f(x_0) \qquad (1)$$

Since the values of x, x_1, x_0, $f(x_1)$, and $f(x_0)$ are known, $f(x)$ can be calculated using Eq. 1.

Yet, in many cases, we may need more complex polynomials make interpolation more accurate.

In general, we can use nth order polynomials for interpolation:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n$$

Langrange Interpolating Polynomials

To approximate a polynomial using k points (x_1, y_1), (x_2, y_2), ... ,(x_k, y_k), we can define Langrange basis $L(x)$ as:

$$L_i(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}$$

Langrage basis functions can be used to generate a polynomial that passes through the points (x_1, y_1), (x_2, y_2), ... ,(x_k, y_k):

$$f(x) = \sum_i y_i L_i(x) \tag{2}$$

Since

$$L_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

we have $f(x_j) = y_j$. Hence, the resulting polynomial in Eqn 2 passes through the points (x_1, y_1), (x_2, y_2), ... ,(x_k, y_k).

Newton's Divided Difference Polynomials

Similar to Lagrange basis, a polynomial can be fitted to the points (x_1, y_1), (x_2, y_2), ... ,(x_k, y_k) using Newton's basis:

$$N_i(x) = \prod_{j=1}^{i-1}(x - x_j)$$

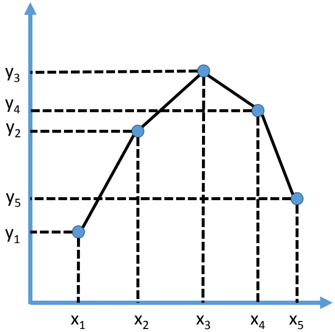
$$f(x) = \sum_{i=1}^{k} c_i N_i(x)$$

Each c_i can be calculated using y_i:

$$\begin{aligned} c_1 &= y_1 \\ c_2 &= \frac{(y_2 - c_1)}{x_2 - x_1} \\ &\ldots \end{aligned} \tag{3}$$

Linear Spline Interpolation

In Linear Spline Interpolation, a line is fitted between two consequtive data points as shown in the figure below.



Between each data point, the equation of the fitted line can be found by:

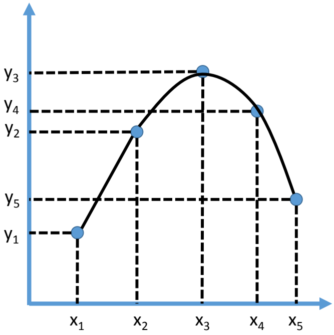
$$f_i(x) = y_i + \frac{y_{i+1} - y_i}{x_{i+1} - x_i}(x - x_i)$$

Although fitting a line can be a good approximation for consequtive samples, in some cases, we may need futher complexity to achieve higher accuracy.

Quadratic Spline Interpolation

Quadratic spline interpolation fits a quadratic polynomial between consequtive data points.

$$y_i = a_i x^2 + b_i x + c_i \tag{4}$$



Eqn. 4 has three unknowns. Using only two samples is not enough to find the parameters of this equation.

One important constraint of quadratic spline interpolation is its continuous derivative constraint. For two consequtive fits, the derivative at their intersection should be the same:

$$\frac{\mathrm{d}}{\mathrm{d}x} a_1 x^2 + b_1 x + c_1 = \frac{\mathrm{d}}{\mathrm{d}x} a_2 x^2 + b_2 x + c_2, \text{at } x = x_2 \quad (5)$$

In addition to this constraint, we need one more equation to find a unique solution. We solve this by adding one more constraint that is the first fit is not quadratic but linear. Hence, we fix a_1 to 0.

Conclusion

- Newton and Langrage basis functions provide a general polynomial based on all given samples which produces reasonable approximations for the purpose of interpolation.
- Compared to Newton and Langrange interpolating Polynomials, Linear Spline Interpolation uses consequtive samples to fit lines between these samples. This approach is computationally efficient however linear assumption may be not adequate for high accuracy interpolation results.
- To have continuous derivatives between the fits and to increase complexity, quadratic splines applies quadratic polynomial fitting between consequtive data samples.