Differential Equations

Lecture 7

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Derivative of a Function:

The derivative of a function at $x = x_0$ is:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

as h gets close to zero (for very small h), we expect round-off errors hamper the result substantially.

Differential Equations

Dependent Variable: The quantity that is been differentiated is called the dependent variable.

Independent Variable: The quantity that is been used for differentiating the dependent variable is called the independent variable.

$$\frac{dv}{dt} = g - \frac{c}{m}v \tag{1}$$

In the equation above, v is the dependent variable and t is the independent variable.

We can classify differential equations into two groups: ordinary differential equation and partial differential equations.

Ordinary Differential Equations (ODE): In an ordinary differential equation, there is only one independent variable as in Eqn. **??**.

Partial Differential Equation (PDE): In contrast to ODEs, we have multiple independent variable in PDEs.

We can also group differential equations with respect to their order, first order, second order, etc.

Higher order equations can be reduced to first order equation by introducing a new variable.

Assume a mass-spring system, let m, k, and c denote the mass, the spring constant and the damping effect, respectively:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \tag{2}$$

In Eqn. **??**, the higher order derivative term is $\frac{d^2x}{dt^2}$. To convert this equation into first order ODE, we introduce a new variable y, such that:

$$y = \frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{d^2x}{dt^2} \tag{3}$$

When we substitude Eqn. **??** into Eqn. **??**:

$$\begin{aligned} m\frac{dy}{dt} + cy + kx = 0 \\ \Rightarrow \frac{dy}{dt} = -\frac{cy + kx}{m} \end{aligned} \tag{4}$$

Hence, we end up with a first order equation as given in Eqn. **??**. We can calculate derivative of a function numerically.

Numerical Derivative:

Taking the derivative analytically may not always be possible or feasible. We rather prefer to take the derivative numerically. There are several methods to achieve this task and in this lecture we will see some of the most common ones.

We need to keep in mind that there is always an error in numerical solutions and if analytical solution exists, we can calculate this error by calculating the difference between the numerical and analytical solutions.

Taylor Series Expansion:

The value of of a function, $f(x)$, at $x = x_0$ can be found using Taylor Series Expansion:

$$f(x_0) = f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

We can express the next number, x_{i+1} in sequence of x_i's by using Taylor Series Expansion as:

$$f(x_{i+1}) = f(x_i) + f^{'}(x_i)h + \frac{f^{(2)}(x_i)}{2!}h + ... \tag{5}$$

From Eqn. **??**, we can calculate $f'(x_i)$ as:

$$f'(x_i) = \frac{1}{h}(f(x_{i+1}) - f(x_i) - \frac{f^{(2)}}{2}h + ...)$$

If we represent the higher order terms as $O(h^2)$:

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i))}{h} - \frac{f^{(2)}(x_i)}{2}h + O(h^2) \tag{6}$$

We can calculate the first derivative as:

$$f^{(1)}(x_i) = \frac{f(x_{i+1}) - f(x_i))}{h} + O(h) \tag{7}$$

Similarly, we can calculate the second derivative as:

$$f^{(2)}(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2} + O(h) \tag{8}$$

We can substitude Eqn. **??** in Eqn. **??**:

$$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} + O(h^2) \qquad (9)$$

In comparison to $O(h)$ in Eqn. **??**, we have $O(h^2)$ in Eqn. **??** which denotes that our error in numerical differentiation gets smaller.