# ENE 206 – Fluid Mechanics

## WEEK 9

- **Conservation of mass**

➢ The application of the principle of conservation of mass to an infinitesimal fluid element results in an equation which is known as the continuity equation.

The principles of conservation of mass requires that the net outflow of mass per unit time from the infinitesimal control volume must be equal to the rate of decrease of mass within the infinitesimal control volume. Then the relevant equations are given as follows:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0 \qquad (6.5)$$

or alternatively:

$$\frac{D\rho}{dt}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0 \qquad (6.6)$$

Equation 6.5 and 6.6 are known as the differential formulation of the continuity equation. The equation can be also in vector notation as:

$$\frac{\partial \rho}{\partial t}+\nabla.\left(\rho\vec{V}\right)=0 \qquad (6.7)$$

and

$$\frac{D\rho}{dt}+\rho.\left(\nabla.\vec{V}\right)=0 \qquad (6.8)$$

For a steady flow,

$$\nabla.\left(\rho\vec{V}\right)=0 \qquad (6.9)$$

For an incompressible fluid flow,

$$\nabla.\vec{V}=0 \qquad (6.10)$$

This equation can be alternatively written as $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$ .

- **The stream function**

A stream function is defined to relate the streamlines which are the imaginary lines describing the fluid flow motion to the principle of conservation of mass. It is possible to define a stream function either for a two-dimensional flow of an incompressible fluid or for a two-dimensional and steady flow.

➢ When Two-dimensional flow of an incompressible fluid:

The continuity equation becomes:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (6.11)$$

Here a continuous function is defined as $\psi = \psi(x, y, t)$, such that

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x} \tag{6.12}$$

- The stream function for a two-dimensional and steady flow

  The continuity equation for a defined *xy*-plane of the Cartesian coordinates becomes:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{6.13}$$

Then it is possible to define a continuous stream function $\psi = \psi(x, y)$ such that

$$\rho u = \frac{\partial \psi}{\partial y} \text{ and } \rho v = -\frac{\partial \psi}{\partial x} \tag{6.14}$$

which also satisfies the continuity equation.

## • Euler's equation

- When the Newton's second law of motion is applied to an infinitesimal fluid element of mass for an inviscid fluid, the resultant equation is the Euler's equation. For deriving the Euler's equation, an infinitesimal fluid element of mass *dm* with sides *dx*, *dy,* and *dz* in the Cartesian Coordinates are defined such that

$$d\vec{F} = a\,dm \tag{6.15}$$

where $d\vec{F}$ is the sum of forces acting on the infinitesimal fluid element and $\vec{a} = D\vec{V}/dt$ is the acceleration of this element. Eqn. 6.15 can be rewritten as follows:

$$d\vec{F} = \rho \frac{D\vec{V}}{dt} dx\,dy\,dz \tag{6.16}$$

There are in general two types of forces acting on a fluid element namely body forces and surface forces. The body force term on the infinitesimal fluid element is:

$$d\vec{F}_b = \rho \vec{f}_b\, dx\,dy\,dz \tag{6.17}$$

where $\vec{f}_b$ is the body force per unit mass.

In an inviscid fluid, there is no shear stress and the only surface force is due to the pressure term. Then the pressure force on the infinitesimal fluid element is:

$$d\vec{F}_s = -\left(\frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}\right) dx\,dy\,dz = -\nabla p\, dx\,dy\,dz \tag{6.18}$$

Then the total force acting on the infinitesimal fluid element is:

$$d\vec{F} = d\vec{F}_s + d\vec{F}_b = \rho \vec{f}_b\, dx\,dy\,dz - \nabla p\, dx\,dy\,dz \tag{6.19}$$

In the final form, eqn. 6.19 becomes:

$$\rho \frac{D\vec{V}}{dt} = \rho \vec{f}_b - \nabla p \tag{6.20}$$

Eqn. 6.20 is known as the Euler's equation.

- **Navier-Stokes equation**

- When the Newton's second law of motion is applied to an infinitesimal fluid element of mass for an viscous fluid, the resultant equation is the Navier-Stokes equation. For deriving the Navier-Stokes equation, an infinitesimal fluid element of mass *dm* with sides *dx*, *dy,* and *dz* in the Cartesian Coordinates are defined such that
- $d\vec{F} = \vec{a}dm$ (6.21)

where $d\vec{F}$ is the sum of forces acting on the infinitesimal fluid element and $\vec{a} = D\vec{V}/dt$ is the acceleration of this element. Eqn. 6.21 can be rewritten as follows:

$$d\vec{F} = \rho\frac{D\vec{V}}{dt}dx\,dy\,dz \tag{6.22}$$

There are in general two types of forces acting on a fluid element namely body forces and surface forces. The body force term on the infinitesimal fluid element is:

$$d\vec{F}_b = \rho\vec{f}_b\,dx\,dy\,dz \tag{6.23}$$

where $\vec{f}_b$ is the body force per unit mass.

In a viscous fluid, there are two types of surface forces which are due to the pressure distribution and the normal and shear distribution on the infinitesimal fluid element. The component of surface force in the Cartesian coordinates can be expressed as follows:

$$dF_{sx} = \left(-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)dx\,dy\,dz \tag{6.24a}$$

$$dF_{sy} = \left(-\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right)dx\,dy\,dz \tag{6.24b}$$

$$dF_{sz} = \left(-\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right)dx\,dy\,dz \tag{6.24c}$$

Hence the components of the force acting on the flluid element in the x, y and z directions can be obtained by combining eqn. 6.23 and 6.24 as:

$$dF_x = dF_{bx} + dF_{sx} = \left(\rho f_{bx} - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)dx\,dy\,dz \tag{6.25a}$$

$$dF_y = dF_{by} + dF_{sy} = \left(\rho f_{by} - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right)dx\,dy\,dz \tag{6.25b}$$

$$dF_z = dF_{bz} + dF_{sz} = \left(\rho f_{bz} - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right)dx\,dy\,dz \tag{6.25c}$$

By expressing eqn. 6.22 in three coordinates and combining these expressions with eqn 6.25, the equations of motion in the *x, y* and *z* directions can take the following form:

$$\rho\frac{Du}{dt} = \rho f_{bx} - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \tag{6.26a}$$

$$\rho\frac{Dv}{dt}=\rho f_{by}-\frac{\partial p}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z} \tag{6.26b}$$

$$\rho\frac{Dw}{dt}=\rho f_{bz}-\frac{\partial z}{\partial z}+\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z} \tag{6.26c}$$

For most of the engineering applications, the variations in the viscosity are neglected. Then the Navier-Stokes equations are greatly simplified for an incompressible fluid case. It is possible to express the Navier-Stokes equations in the vectorial notation as:

$$\rho\frac{D\vec{V}}{dt}=-\nabla p+\mu\nabla^2\vec{V}+\rho\vec{f}_b \tag{6.27}$$

- ## References

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