Example-2

For the region S in Example 1, show that the sum of the areas of the upper approximating rectangles approaches $\frac{1}{3}$, that is

$$\lim_{n\to\infty} R_n = \frac{1}{3}$$

Solution

R_n is the sum of the areas of the n rectangles in Figure 7.

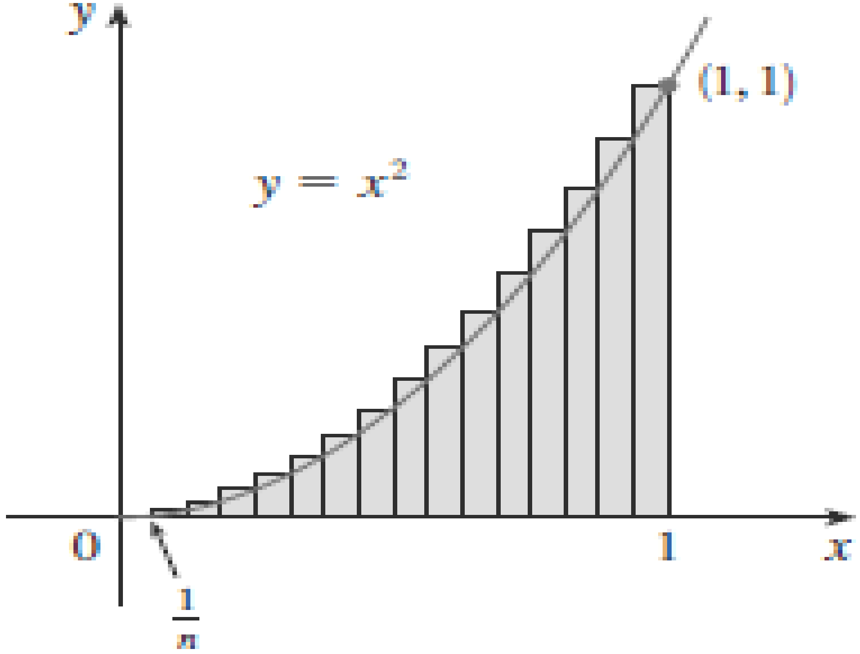

Figure 7

From Figures 8 and 9 it appears that, as n increases, both L_n and R_n become better and better approximations to the area of S.

Therefore, we define the area A to be the limit of the sums of the areas of the approximating rectangles, that is,

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} L_n = \frac{1}{3}$$

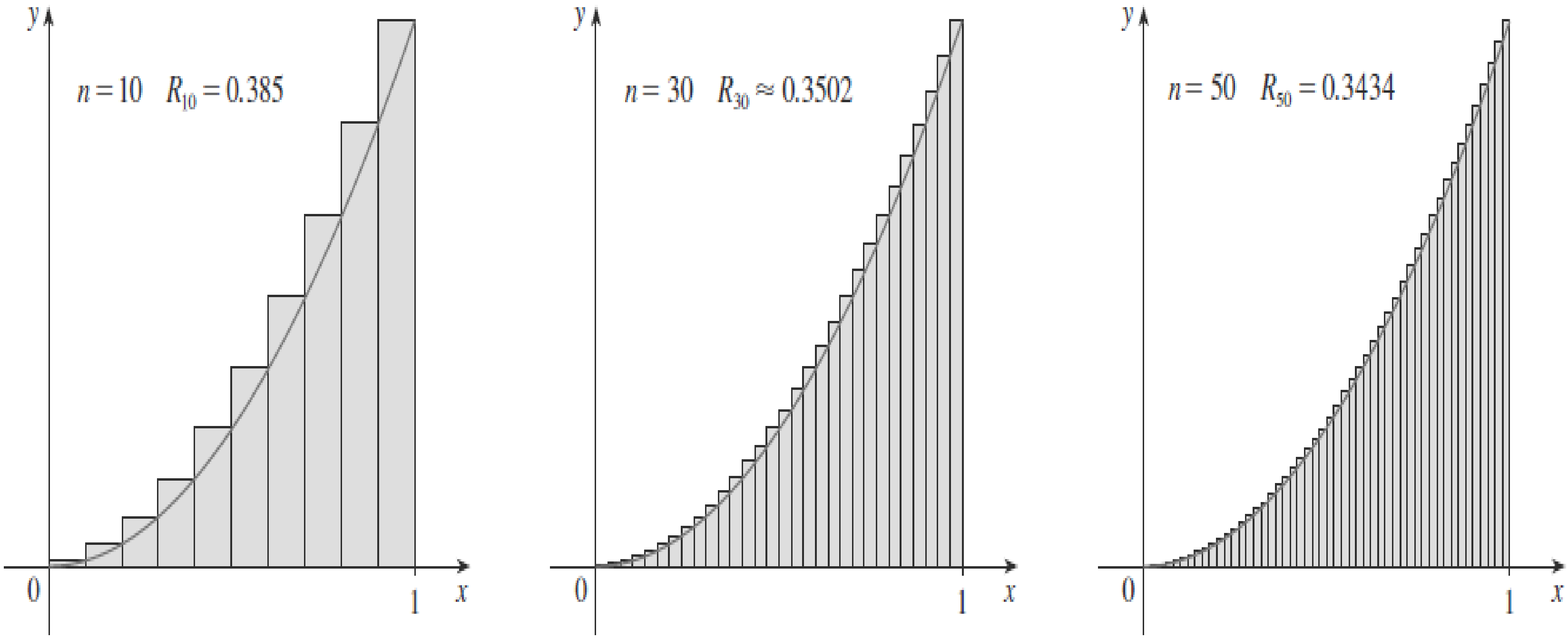
y
$n = 10$ $R_{10} = 0.385$
0
1
x
y
$n = 30$ $R_{30} \approx 0.3502$
0
1
x
y
$n = 50$ $R_{50} = 0.3434$
0
1
x

FIGURE 8

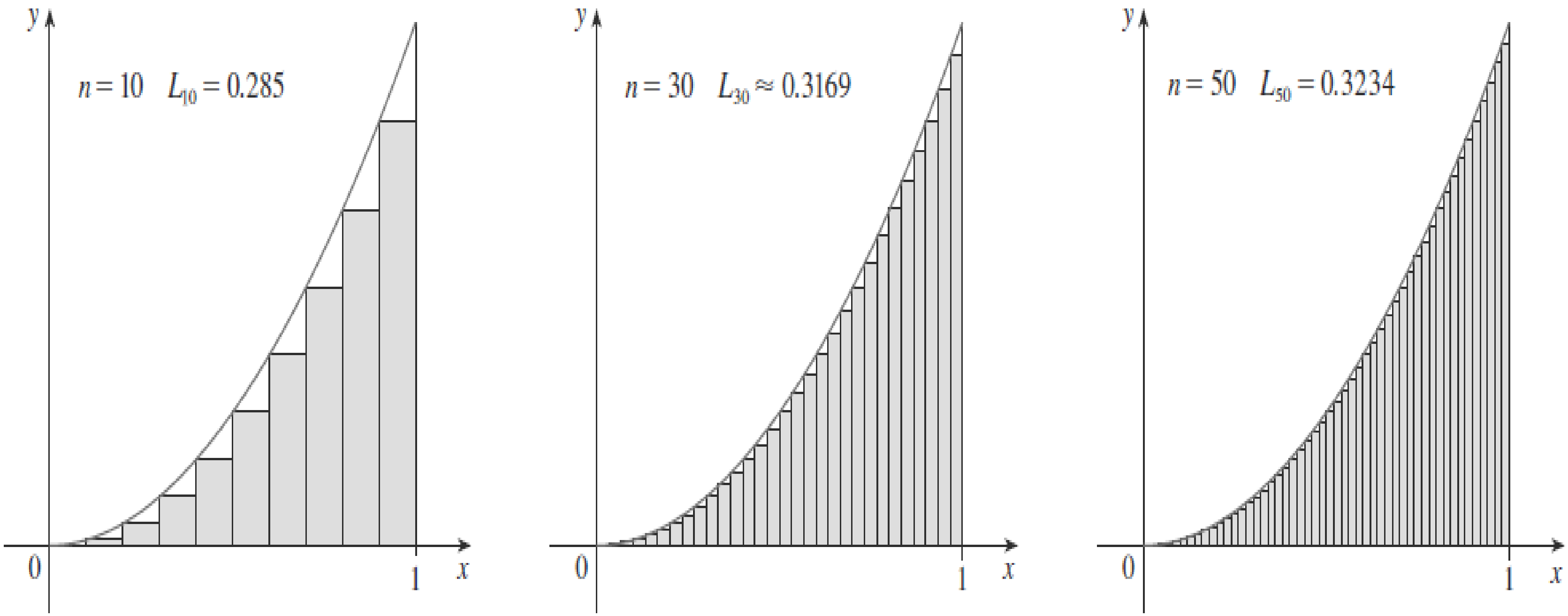

FIGURE 9
The area is the number that is smaller than all upper sums and larger than all lower sums

Let's apply the idea of Examples 1 and 2 to the more general region S of Figure 1. We start by subdividing S into n strips $S_1, S_2, \dots, S_n$ of equal width as in Figure 10.

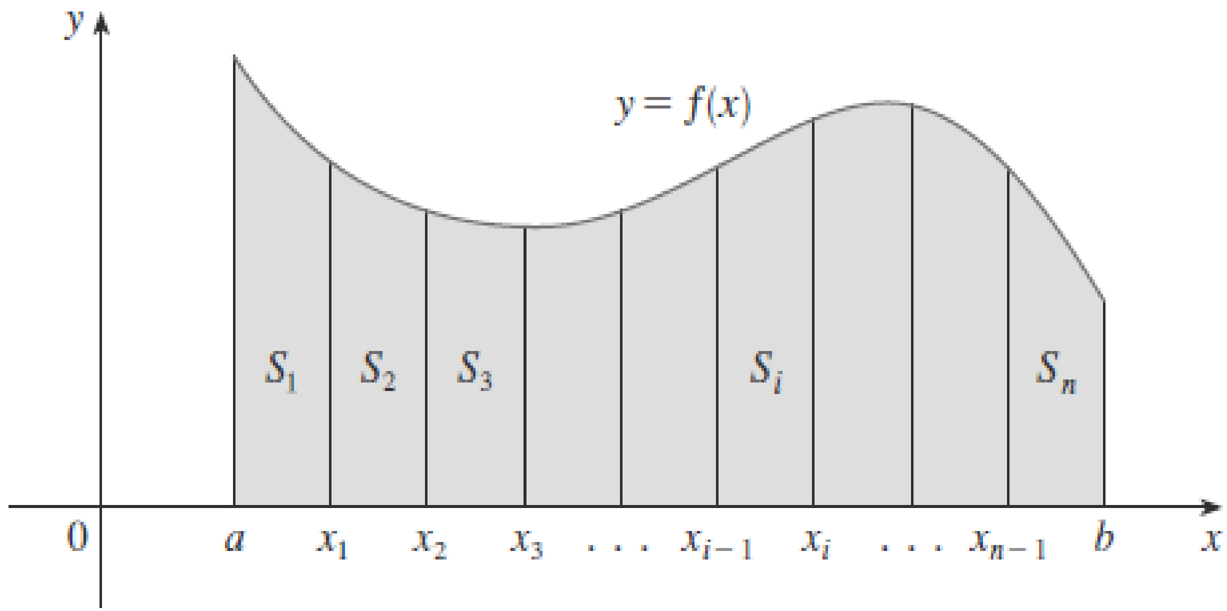

Figure 10

The width of the interval $[a, b]$ is $b - a$, so the width of each of the n strips is

$$\Delta x = \frac{b - a}{n} \ .$$

These strips divide the interval [a, b] into n subintervals

$$[x_0, x_1], [x_1, x_2], \ [x_2, x_3], \ \ldots, \ [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$.

The right endpoints of the subintervals are

$x_1 = a + \Delta x,\ x_2 = a + 2\Delta x,\ \ x_3 = a + 3\Delta x, \ldots$

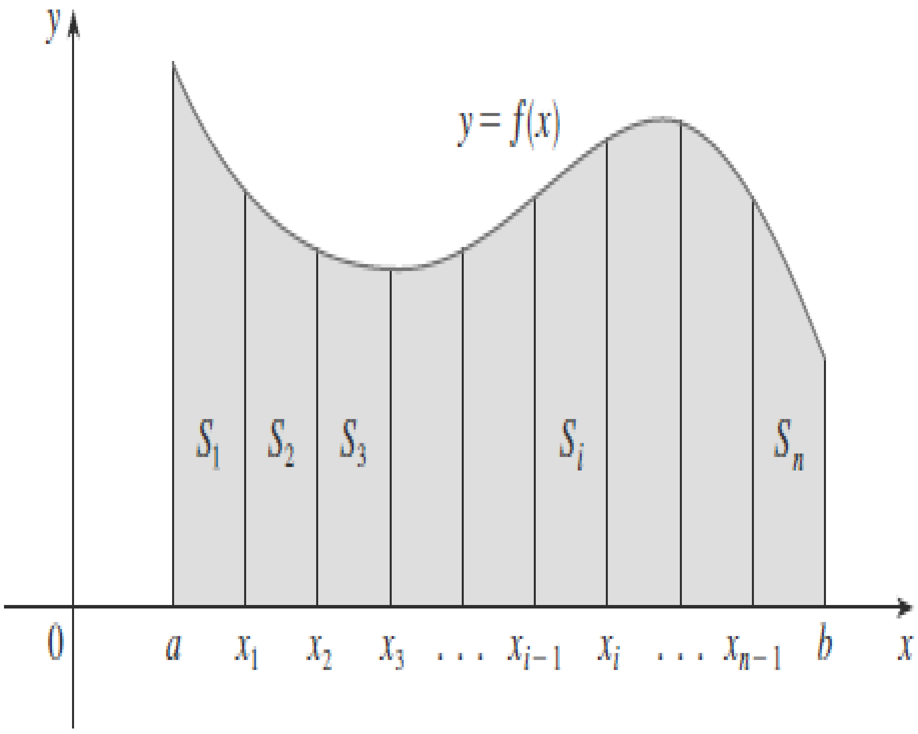

FIGURE 10

Let's approximate the ith strip S_i by a rectangle with width Δx and height $f(x_i)$, which is the value of f at the right endpoint (see Figure 11).

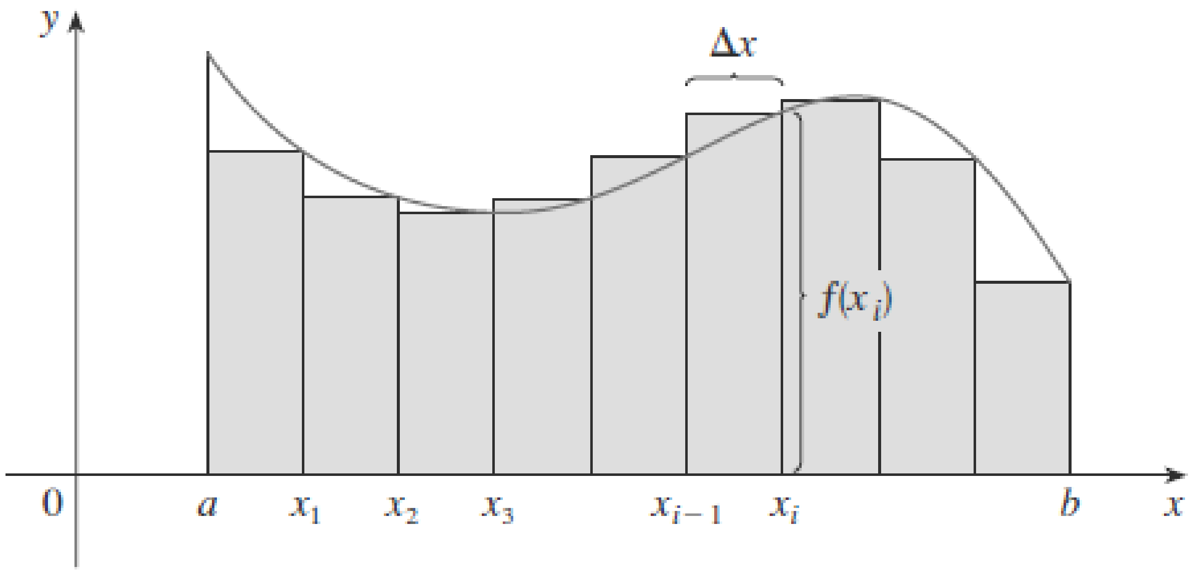

Figure 11

Then the area of the ith rectangle is $f(x_i)\Delta x$.

What we think of intuitively as the area of S is approximated by the sum of the areas of these rectangles, which is

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

Figure 12 shows this approximation for n 2, 4, 8, and 12. Notice that this approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$.

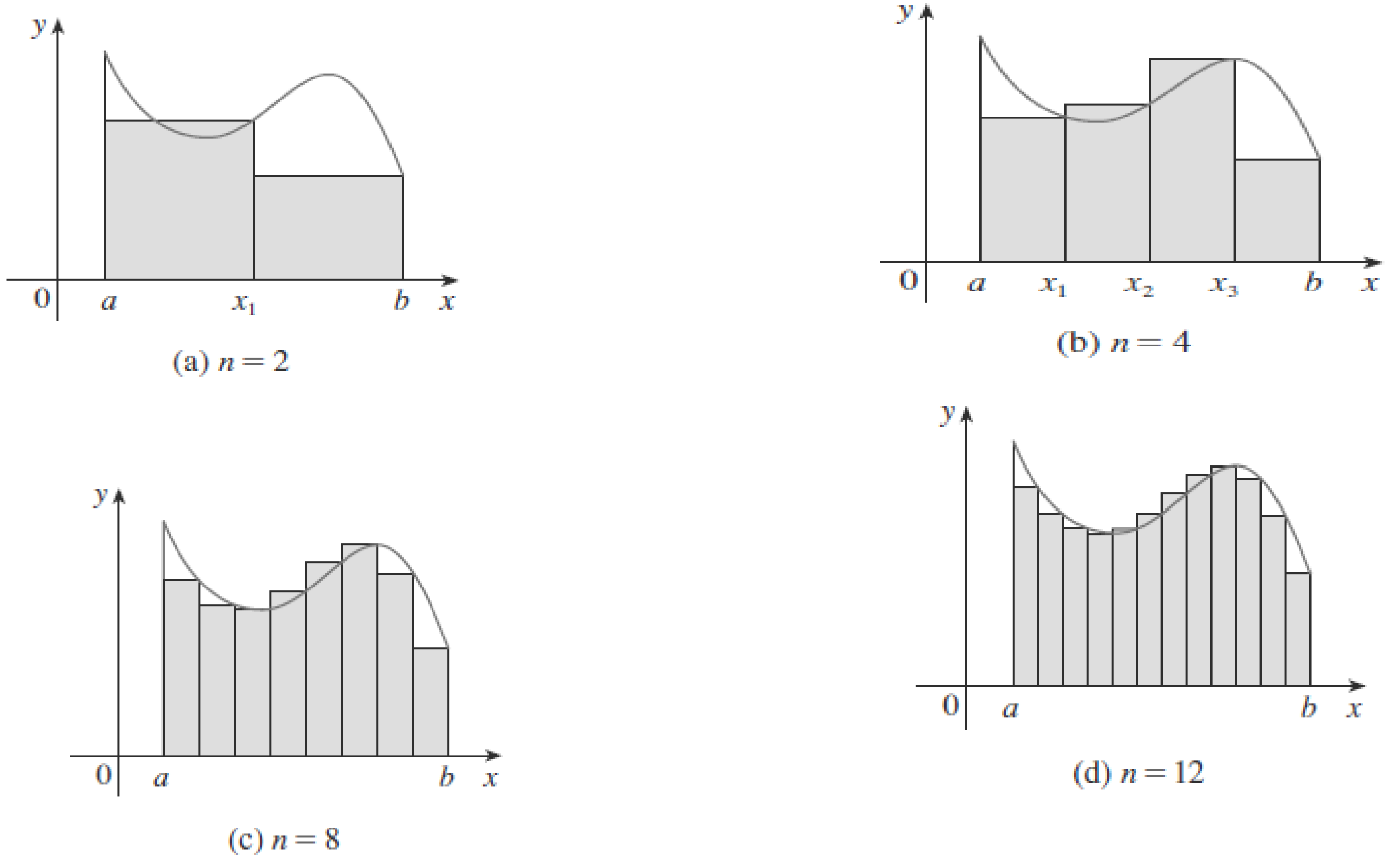

Figure 12

Therefore, we define the area A of the region S in the following way.

Definition 1. The **area** A of the region S that lies under the graph of the continuous function is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x] \ (1)$$

It can be proved that the limit in Definition 1 always exists, since we are assuming that is continuous.

It can also be shown that we get the same value if we use left endpoints:

$$A = \lim_{n\to\infty} L_n$$
$$= \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x] \quad (2)$$

In fact, instead of using left endpoints or right endpoints, we could take the height of the ith rectangle to be the value of f at ***any*** number x_i^* in the ith subinterval $[x_{i-1}, x_i]$.

We call the numbers $x_1^*, x_2^*, \ldots, x_n^*$ **the sample points.**

Figure 13 shows approximating rectangles when the sample points are not chosen to be endpoints.

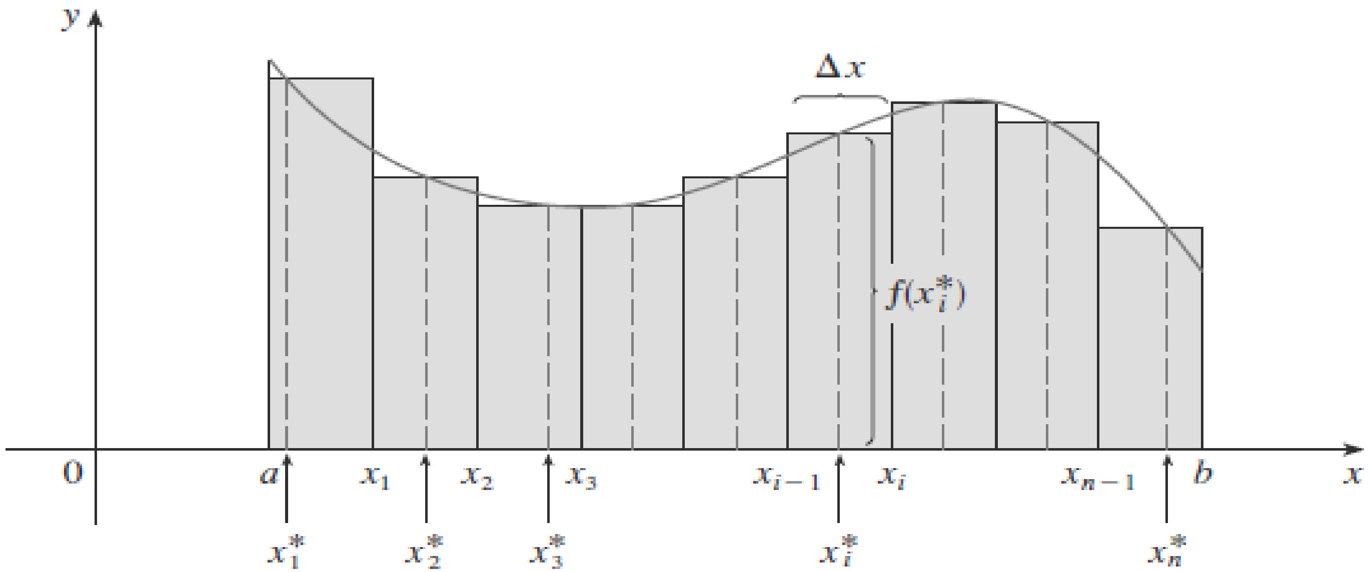

Figure 13

So a more general expression for the area of S is

$$A = \lim_{n\to\infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x] \quad (3)$$

We often use sigma notation to write sums with many terms more compactly. For instance,

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

So the expressions for area in Equations 1, 2, and 3 can be written as follows:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$