Example-1

Express

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(x_i^3 + x_i \sin x_i \right) \Delta x$$

as an integral on the interval $[0, \pi]$.

Evaluating Integrals

When we use the definition to evaluate a definite integral, we need to know how to work with sums.

The following three equations give formulas for sums of powers of positive integers.

Equation 4 may be familiar to you from a course in algebra. Equations 5 and 6 were discussed in Section 1.1.

$$\sum_{i=1}^{n} i = \frac{n\,(n+1)}{2} \tag{4}$$

$$\sum_{i=1}^{n} i^2 = \frac{n\,(n+1)(2n+1)}{2} \tag{5}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n\,(n+1)}{2}\right]^2 \tag{6}$$

The remaining formulas are simple rules for working with sigma notation:

$$\sum_{i=1}^{n} c = nc \tag{7}$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i \tag{8}$$

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \tag{9}$$

$$\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \tag{10}$$

Example-2

(a) Evaluate the Riemann sum for $f(x) = x^3 - 6x$ taking the sample points to be right endpoints and $a = 0, \mathrm{b} = 3$ and $\mathrm{n} = 6$.

(b) Evaluate $\int_0^3 x^3 - 6x\, dx$.

Solution

(a)

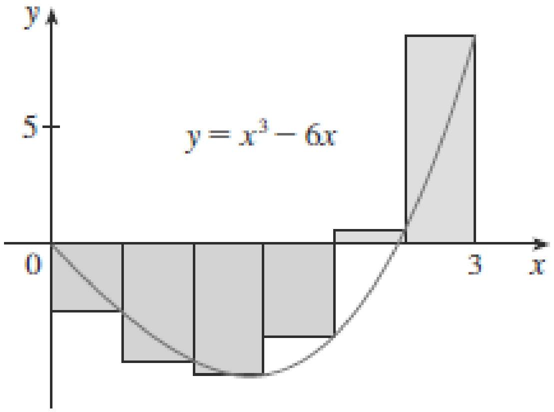

FIGURE 5

Notice that f is not a positive function and so the Riemann sum does not represent a sum of areas of rectangles. But it does represent the sum of the areas of the **above the *x*-axis minus the sum of the areas of below the *x*-axis** in Figure 5.

(b)

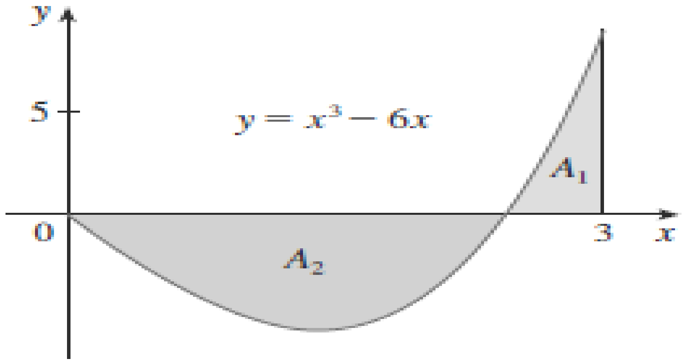

FIGURE 6
$\int_0^3 (x^3 - 6x)\,dx = A_1 - A_2 = -6.75$

This integral can't be interpreted as an area because f takes on both positive and negative values. But it can be interpreted as the difference of areas , $A_1 - A_2$ where A_1 and A_2 are shown in Figure 6.

Figure 7 illustrates the calculation by showing the positive and negative terms in the right Riemann sum R_n for $n = 40$. The values in the table show the Riemann sums approaching the exact value of the integral, -6.75, as $n \to \infty$.

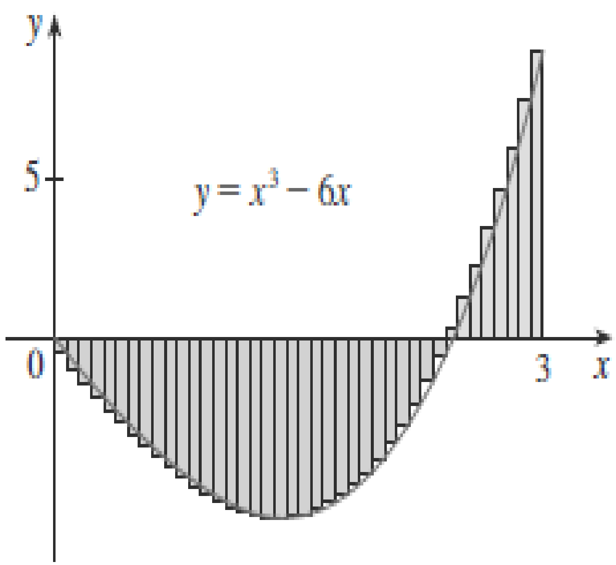

FIGURE 7
$R_{40} \approx -6.3998$

n	R_n
40	−6.3998
100	−6.6130
500	−6.7229
1000	−6.7365
5000	−6.7473

Example-3

Set up an expression for $\int_1^3 e^x \, dx$ as a limit of sums.

Solution

IIII Because $f(x) = e^x$ is positive, the integral in Example 3 represents the area shown in Figure 8.

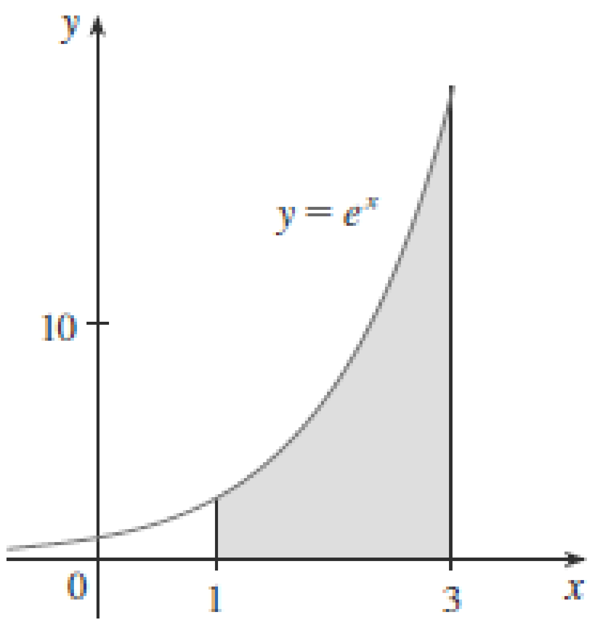

FIGURE 8

Example-4

Evaluate the following integrals by interpreting each in terms of areas.

(a) $\int_0^1 \sqrt{1-x^2}\ dx$ (b) $\int_0^3 (x-1)\, dx$

Solution

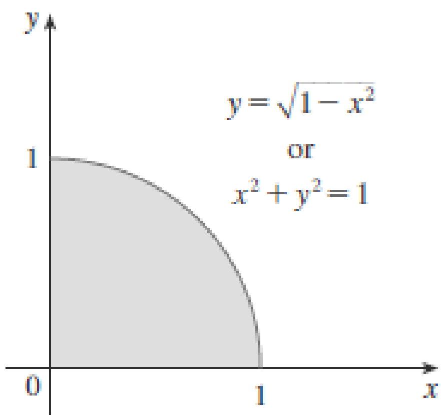

FIGURE 9

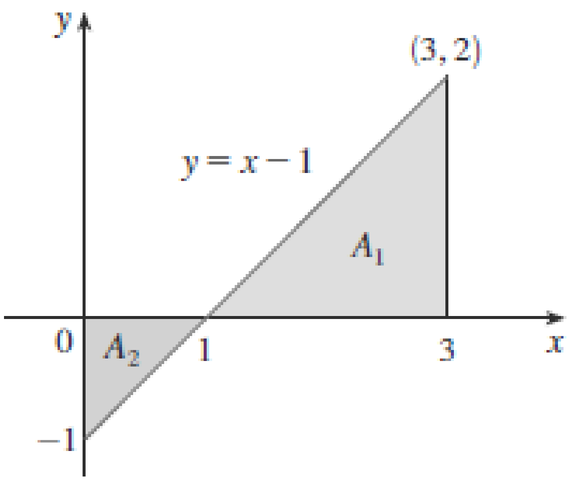

Figure 10

The Midpoint Rule

We often choose the sample point x_i^* to be the right endpoint of the th subinterval because it is convenient for computing the limit. But if the purpose is to find an approximation to an integral, it is usually better to choose x_i^* to be the midpoint of the interval, which we denote by $\overline{x_i}$. Any Riemann sum is an approximation to an integral, but if we use midpoints we get the following approximation.

Midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\overline{x_i})\Delta x = \Delta x[f(\overline{x_1}) + f(\overline{x_2}) + \cdots + f(\overline{x_n})$$

where

$$\Delta x = \frac{b-a}{n}$$

and

$$\overline{x_i} = \frac{1}{2}(x_{i-1} + x_i) = midpoint\ of\ [x_{i-1}, x_i]$$

Example-5

Use the Midpoint Rule with n = 5 to approximate

$$\int_1^2 \frac{1}{x}\,dx$$

Solution

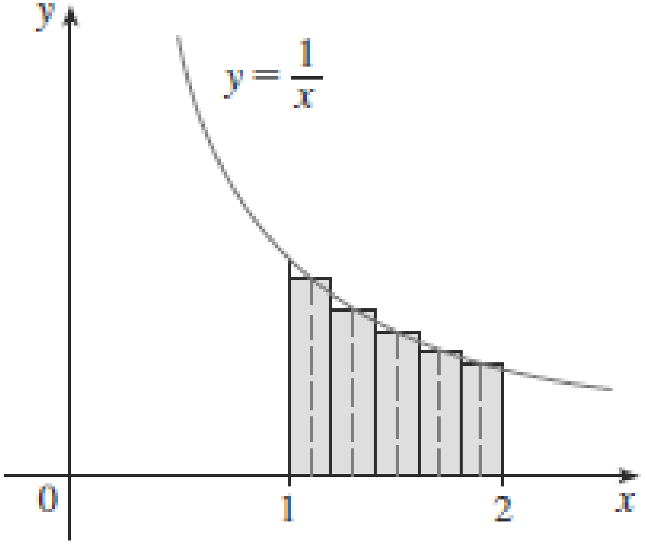

FIGURE 11

If we apply the Midpoint Rule to the integral in Example 2, we get the picture in Figure 12. The approximation $M_{40} \approx -6.7563$ is much closer to the true value -6.75 than the right endpoint approximation, $R_{40} \approx -6.3998$, shown in Figure 7.

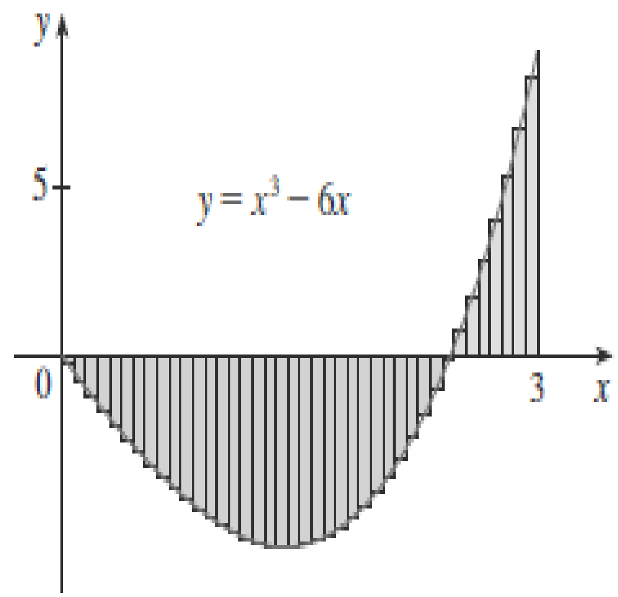

FIGURE 12
$M_{40} \approx -6.7563$

Properties of the Definite Integrals

When we defined the definite integral $\int_a^b f(x)dx$, we implicitly assumed that $a < b$. But the definition as a limit of Riemann sums makes sense even if $a > b$. Notice that if we reverse a and b, then Δx changes from $(b-a)/n$ to $(a-b)/n$. Therefore

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$