Plane waves in a lossy medium:
In a lossy medium, the EM wave loses power as it propagates. Such a medium is conducting with conductivity σ and we can write:

$$\nabla \times \vec{H} = \vec{J} + j\omega\vec{D}$$

$$\nabla \times \vec{H} = \sigma\vec{E} + j\omega\boldsymbol{\varepsilon}\vec{E}$$

$$\nabla \times \vec{H} = (\sigma + j\omega\boldsymbol{\varepsilon})\vec{E}$$

$$\nabla \times \vec{H} = j\omega(\boldsymbol{\varepsilon} + \frac{\sigma}{j\omega})\vec{E}$$

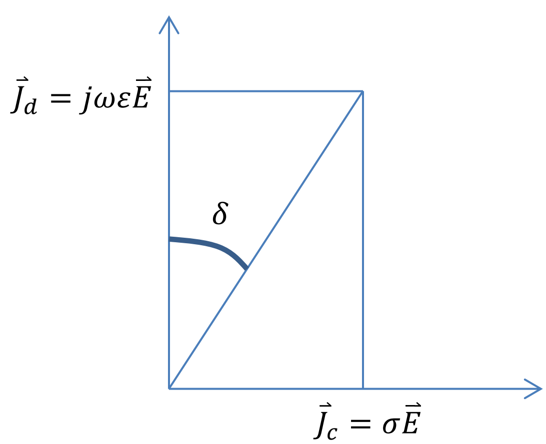
$$\nabla \times \vec{H} = j\omega\varepsilon_c\vec{E}$$

Where $\varepsilon_c = \varepsilon - j\dfrac{\sigma}{\omega} = \varepsilon' - j\varepsilon''$ is called the complex permittivity.

An external electric field can polarize a dielectric and give rise to bound charges. When the external electric field is time varying, the polarization vector will vary with the same frequency as that of the applied field. As the frequency of the applied filed increases, the inertia of the charge particles tend to prevent the particle displacement keeping pace with the applied field changes. This results in frictional damping mechanism causing power loss.

In addition, if the material has an appreciable amount of free charges, there will be ohmic losses. The effect of damping and ohmic losses are represented in the imaginary part of ε_c. An equivalent conductivity $\sigma = \omega\varepsilon''$ represents all losses.

The ratio $\dfrac{\varepsilon''}{\varepsilon'}$ is called loss tangent as this quantity is a measure of the power loss.



$$tan\delta = \frac{|\vec{J}_c|}{|\vec{J}_d|} = \frac{\sigma}{\omega\varepsilon} = \frac{\varepsilon''}{\varepsilon'}$$

Where $\vec{J}_c$ is the conduction current density and $\vec{J}_d$ is displacement current density. The loss tangent gives a measure of how much lossy is the medium under consideration. For a good dielectric medium $(\sigma << \omega\varepsilon) \tan\delta$ is very small and the medium is a good conductor at low frequencies but behave as lossy dielectric at higher frequencies.

For a source free lossy medium we can write

$$\nabla \times \vec{H} = (\sigma + j\omega\varepsilon)\vec{E}$$

$$\nabla \times \vec{E} = -j\omega\vec{B} = -j\omega\mu\vec{H}$$

$$\nabla \bullet \vec{E} = 0$$

$$\nabla \bullet \vec{H} = 0$$

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \bullet \vec{E}) - \nabla^2\vec{E}$$
$$\nabla \times \nabla \times \vec{E} = \nabla \times (-j\omega\mu\vec{H})$$
$$\nabla(\nabla \bullet \vec{E}) - \nabla^2\vec{E} = (-j\omega\mu)(\sigma + j\omega\varepsilon)\vec{E}$$

$$\nabla^2\vec{E} - (j\omega\mu)(\sigma + j\omega\varepsilon)\vec{E} = 0$$

$$\gamma^2 = (j\omega\mu)(\sigma + j\omega\varepsilon)$$

$$\nabla^2\vec{E} - \gamma^2\vec{E} = 0$$

Proceeding in the same manner we can write,

$$\nabla^2\vec{H} - \gamma^2\vec{H} = 0$$

$$\gamma = \alpha + i\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = j\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)^{1/2}$$

is called the propagation constant.

The real and imaginary parts α and β of the propagation constant γ can be computed as follows:

$$\gamma^2 = (\alpha + i\beta)^2 = j\omega\mu(\sigma + j\omega\varepsilon)$$

$$or, \alpha^2 - \beta^2 = -\omega^2\mu\varepsilon$$

$$\text{And } \alpha\beta = \frac{\omega\mu\sigma}{2}$$

$$\therefore \alpha^2 - \left(\frac{\omega\mu\sigma}{2\alpha}\right)^2 = -\omega^2\mu\varepsilon$$

$$or, 4\alpha^4 + 4\alpha^2\omega^2\mu\varepsilon = \omega^2\mu^2\sigma^2$$

$$or, 4\alpha^4 + 4\alpha^2\omega^2\mu\varepsilon + \omega^4\mu^2\varepsilon^2 = \omega^2\mu^2\sigma^2 + \omega^4\mu^2\varepsilon^2$$

$$or, \left(2\alpha^2 + \omega^2\mu\varepsilon\right)^2 = \omega^4\mu^2\varepsilon^2\left(1 + \frac{\sigma^2}{\omega^2\varepsilon^2}\right)$$

$$or, \alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]}$$

$$\text{Similarly, } \beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]}$$

Let us now consider a plane wave that has only x-component of electric field and propagate along z.

$$\vec{E}_x(z) = (E_0^+ e^{-\gamma z} + E_0^- e^{+\gamma z})\hat{x}$$

Considering only the forward traveling wave

$$\vec{E}_x(z,t) = Re\left\{E_0^+ e^{-\gamma z} e^{j\omega t}\right\}\hat{x}$$

$$\vec{E}_x(z,t) = \cos(\omega t - \beta z)\,\hat{x}$$

Similarly, from

$$\vec{H}(R) = \frac{1}{\eta}(\hat{n} \times \vec{E}_0 e^{-\gamma \cdot \vec{R}})$$

Where $\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |n| e^{j\theta_n}$

$$\vec{H}(R) = \frac{1}{|\eta| e^{j\phi_\eta}} (\hat{n} \times \vec{E}_0 e^{-\gamma \cdot \vec{R}})$$

$$\vec{H}_y(z,t) = Re\{H_0^+ e^{-\gamma z} e^{j\omega t}\} \hat{y}$$

$$\vec{H}_y(z,t) = \frac{|\vec{E}_0|}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \phi_\eta + \phi_{E_0}) \hat{x}$$

From (25) and (26) we find that as the wave propagates along z, it decreases in amplitude by a factor $e^{-\alpha z}$. Therefore α is known as attenuation constant. Further, if $\vec{E}_0$ is real, ***E*** and ***H*** are out of phase by an angle θ_n.

For low loss dielectric, $\frac{\sigma}{\omega\varepsilon} << 1$ i.e., $\varepsilon'' << \varepsilon'$.

Using the above condition approximate expression for α and β can be obtained as follows:

$$\gamma = \alpha + i\beta = j\omega\sqrt{\mu\varepsilon'}\left[1 - j\frac{\varepsilon''}{\varepsilon'}\right]^{1/2}$$

$$\cong j\omega\sqrt{\mu\varepsilon'}\left[1 - j\frac{1}{2}\frac{\varepsilon''}{\varepsilon'} + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right]$$

$$\alpha = \frac{\omega\varepsilon''}{2}\sqrt{\frac{\mu}{\varepsilon'}}$$

$$\& \ \beta = \omega\sqrt{\mu\varepsilon'}\left[1 + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right]$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon'}}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2}$$

$$; \ \sqrt{\frac{\mu}{\varepsilon'}}\left(1 + j\frac{\varepsilon''}{2\varepsilon'}\right)$$

& phase velocity

$$v_p = \frac{\omega}{\beta} \cong \frac{1}{\sqrt{\mu\varepsilon'}}\left[1 - \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right]$$

For good conductors $\frac{\sigma}{\omega\varepsilon} >> 1$

$$\gamma = j\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma}{j\omega\varepsilon}\right) \cong j\omega\sqrt{\mu\varepsilon}\sqrt{\frac{\sigma}{j\omega\varepsilon}}$$

$$= \frac{1+j}{\sqrt{2}}\sqrt{\omega\mu\sigma}$$

We have used the relation

$$\sqrt{j} = \left(e^{j\pi/2}\right)^{1/2} = e^{j\pi/4} = \frac{1}{\sqrt{2}}(1+j)$$

We can write

$$\alpha + i\beta = \sqrt{\pi f \mu\sigma} + j\sqrt{\pi f \mu\sigma}$$

$$\therefore \alpha = \beta = \sqrt{\pi f \mu\sigma}$$

$$\eta = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)}}$$

$$\cong \sqrt{\frac{\mu}{\varepsilon}\frac{j\omega\varepsilon}{\sigma}} = \sqrt{\frac{j\omega\mu}{\sigma}}$$

$$= (1+j)\sqrt{\frac{\pi f \mu}{\sigma}}$$

$$= (1+j)\frac{\alpha}{\sigma}$$

And phase velocity

$$v_p = \frac{\omega}{\beta} \cong \sqrt{\frac{2\omega}{\mu\sigma}}$$