2.5 SUBSTITUTIONS AND TRANSFORMATIONS

When the equation

$$M(x,y)dx + N(x,y)dy = 0$$

is not a separable, exact, or linear equation, it may still be possible to transform it into one that we know how to solve.

This was in fact our approach in Section 2.4, where we used an integrating factor to transform our original equation into an exact equation.

In this section we study four types of equations that can be transformed into either a separable or linear equation by means of a suitable substitution or transformation.

Substitution Procedure

(a) Identify the type of equation and determine the appropriate substitution or transformation.
(b) Rewrite the original equation in terms of new variables.
(c) Solve the transformed equation.
(d) Express the solution in terms of the original variables.

2.5.1 Homogeneous Equations

Definition

If the right-hand side of the equation

(1) $$\frac{dy}{dx} = f(x, y)$$

can be expressed as a function of the ratio y/x alone, then we say the equation is **homogeneous.**

For example, the equation

$$(x - y)dx + x\,dy = 0 \tag{2}$$

can be written in the form

$$\frac{dy}{dx} = \frac{y - x}{x} = \frac{y}{x} - 1 .$$

Since we have expressed $(y - x)/x$ as a function of the ratio y/x $\left[\text{that is, } (y - x)/x = G(y/x),\right.$ where $\left.G(v) := v - 1\right]$, then equation (2) is homogeneous.

The equation

$$(x - 2y + 1)dx + (x - y)dy = 0 \tag{3}$$

can be written in the form

$$\frac{dy}{dx} = \frac{x - 2y + 1}{y - x} = \frac{1 - 2(y/x) + (1/x)}{(y/x) - 1} .$$

Here the right-hand side cannot be expressed as a function of y/x alone because of the term $1/x$ in the numerator. Hence, equation (3) is not homogeneous.

To solve a homogeneous equation, we make a rather obvious substitution. Let

$$v = \frac{y}{x} .$$

Our homogeneous equation now has the form

$$\frac{dy}{dx} = G(v) , \tag{4}$$

and all we need is to express dy/dx in terms of x and v. Since $v = y/x$, then $y = vx$. Keeping in mind that both v and y are functions of x, we use the product rule for differentiation to deduce from $y = vx$ that

$$\frac{dy}{dx} = v + x\frac{dv}{dx} .$$

We then substitute the above expression for dy/dx into equation (4) to obtain

$$v + x\frac{dv}{dx} = G(v) . \tag{5}$$

The new equation (5) is separable, and we can obtain its implicit solution from

$$\int \frac{1}{G(v) - v} dv = \int \frac{1}{x} dx .$$

All that remains to do is to express the solution in terms of the original variables x and y.

Solve Questions

2.5.2 Bernoulli Equations

Definition

A first-order equation that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n ,$$

where and $P(x)$ and $Q(x)$are continuous on an interval (a, b) and n is a real number, is called a **Bernoulli equation.**

Notice that when $n = 0\ or\ 1$, the above equation is also a linear equation and can be solved by the method discussed in Section 2.2. For other values of n, the substitution

$$v = y^{1-n}$$

transforms the Bernoulli equation into a linear equation, as we now show.

Solve Questions

2.5.3 Equations with Linear Coefficients

We have used various substitutions for y to transform the original equation into a new equation that we could solve.

In some cases we must transform both x and y into new variables, say, u and v.

This is the situation for equations with **linear coefficients**-that is, equations of the form

$$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0 \,, \qquad (6)$$

where the a_i's, b_i's, and c_i's are constants.

Before considering the general case when $a_1b_2 \neq a_2b_1$, let's first look at the special situation when $c_1 = c_2 = 0$. Equation 6 then becomes

$$(a_1x + b_1y)dx + (a_2x + b_2y)dy = 0 \ ,$$

which can be rewritten in the form

$$\frac{dy}{dx} = -\frac{a_1x + b_1y}{a_2x + b_2y} = -\frac{a_1 + b_1(y/x)}{a_2 + b_2(y/x)} \ .$$

This equation is homogeneous, so we can solve it using the method discussed earlier in this section.

The above discussion suggests the following procedure for solving (6):

If $a_1b_2 \neq a_2b_1$, then we seek a translation of axes of the form

$$x = u + h \quad \text{and} \quad y = v + k\ ,$$

where h and k are constants, that will change $a_1x + b_1y + c_1$ into $a_1u + b_1v$ and change $a_2x + b_2y + c_2$ into $a_2u + b_2v$. Some elementary algebra shows that such a transformation exists if the system of equations

$$a_1h + b_1k + c_1 = 0\ ,$$
$$a_2h + b_2k + c_2 = 0$$

has a solution.

This means that (h, k) is the intersection point of the lines which are the coefficients of dx and dy.

Now if (h, k) is the intersection point, then the substitution

$$x = u + h \;\; and \; y = v + k$$

transform Eq.(6) into homogeneous equation

$$\frac{dv}{du} = -\frac{a_1 u + b_1 v}{a_2 u + b_2 v} = -\frac{a_1 + b_1(v/u)}{a_2 + b_2(v/u)},$$

Which we know how to solve from Section 2.5.1.

Solve Questions