

### 3. DIFFERENTIAL EQUATIONS AS MODELS

In this section we introduce the notion of a differential equation as a mathematical model and discuss some specific models in biology, chemistry, and physics.

#### MATHEMATICAL MODELS

It is often desirable to describe the behavior of some real-life system or phenomenon, whether physical, sociological, or even economic, in mathematical terms.

The mathematical description of a system of phenomenon is called a **mathematical model** and is constructed with certain goals in mind.

For example,

› we may wish to understand the mechanisms of a certain ecosystem by studying the growth of animal populations in that system,

or

› we may wish to date fossils by analyzing the decay of a radioactive substance either in the fossil or in the stratum in which it was discovered.

Construction of a mathematical model of a system starts with

- I. identification of the variables that are responsible for changing the system. We may choose not to incorporate all these variables into the model at first. In this step we are specifying the **level of resolution** of the model.
- II. we make a set of **reasonable** assumptions, or hypotheses, about the system we are trying to describe. These assumptions will also include any empirical laws that may be applicable to the system.

For some purposes it may be perfectly within reason to be content with low resolution models.

For example, you may already be aware that in beginning physics courses, the retarding force of air friction is sometimes ignored in modeling the motion of a body falling near the surface of the Earth, but if you are a scientist whose job it is to accurately predict the flight path of a long-range projectile, you have to take into account air resistance and other factors such as the curvature of the Earth.

Since the assumptions made about a system frequently involve a ***rate of change*** of one or more of the variables, the mathematical depiction of all these assumptions may be one or more equations involving ***derivatives***.

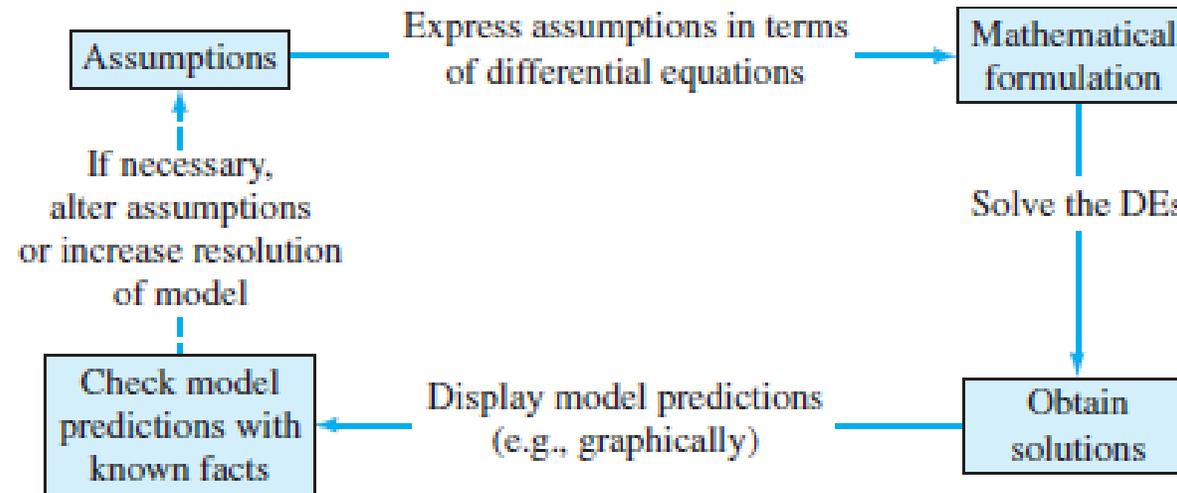
**In other words, the mathematical model may be a differential equation or a system of differential equations.**

Once we have formulated a mathematical model that is either a differential equation or a system of differential equations, we are faced with the not insignificant problem of trying to solve it.

If we can solve it, then we deem the model to be reasonable if its solution is consistent with either experimental data or known facts about the behavior of the system.

But if the predictions produced by the solution are poor, we can either increase the level of resolution of the model or make alternative assumptions about the mechanisms for change in the system.

The steps of the modeling process are then repeated, as shown in the following diagram:



Of course, by increasing the resolution, we add to the **complexity** of the mathematical model and increase the likelihood that we cannot obtain an explicit solution.

A mathematical model of a physical system will often involve the **variable time  $t$** .

A solution of the model then gives **the state of the system**; in other words,

**the values of the dependent variable (or variables) for appropriate values of  $t$  describe the system in the past, present, and future.**

# POPULATION DYNAMICS

One of the earliest attempts to model human population growth by means of mathematics was by the English economist **Thomas Malthus** in 1798.

- › Basically, the idea behind the Malthusian model is the assumption that the rate at which the population of a country grows at a certain time is **proportional** (*This means that one quantity is a constant multiple of the other:  $u = kv$* ) to the total population of the country at that time.
- › In other words, the more people there are at time  $t$ , the more there are going to be in the future.

$\pi$ 

In mathematical terms, if  $P(t)$  denotes the total population at time  $t$ , then this assumption can be expressed as

$$\frac{dP}{dt} = kP \quad (1)$$

where  $k$  is a constant of proportionality.

This simple model, which **fails** to take into account many factors that can influence human populations to either grow or decline (immigration and emigration, for example), nevertheless turned out to be fairly accurate in predicting the population of the United States during the years 1790–1860.

Populations that grow at a rate described by (1) are rare; nevertheless, (1) is still used to model *growth of small populations over short intervals of time* (bacteria growing in a petri dish, for example).

# RADIOACTIVE DECAY

The nucleus of an atom consists of combinations of protons and neutrons. Many of these combinations of protons and neutrons are unstable-that is, the atoms decay or transmute into atoms of another substance. Such nuclei are said to be radioactive.

For example, over time the highly radioactive radium, Ra-226, transmutes into the radioactive gas radon, Rn-222. To model the phenomenon of radioactive decay, it is assumed that the rate  $dA/dt$  at which the nuclei of a substance decay is proportional to the amount (more precisely, the number of nuclei)  $A(t)$  of the substance remaining at time  $t$ :

$$\frac{dA}{dt} = kA \quad (2)$$

Of course, equations (1) and (2) are exactly the same; the difference is only in the interpretation of the symbols and the constants of proportionality. For growth, as we expect in (1),  $k > 0$ , and for decay, as in (2),  $k < 0$ .

- › The model (1) for growth can also be seen as the equation  $\frac{dS}{dt} = rS$ , which describes the growth of capital  $S$  when an annual rate of interest  $r$  is compounded continuously.
- › The model (2) for decay also occurs in biological applications such as determining the half-life of a drug —the time that it takes for 50% of a drug to be eliminated from a body by excretion or metabolism.
- › In chemistry the decay model (2) appears in the mathematical description of a first-order chemical reaction.

The point is this:

*A single differential equation can serve as a mathematical model for many different phenomena.*

Mathematical models are often accompanied by certain side conditions. For example, in (1) and (2) we would expect to know, in turn, the initial population  $P_0$  and the initial amount of radioactive substance  $A_0$  on hand.

If the initial point in time is taken to be  $t = 0$ , then we know that  $P(0) = P_0$  and  $A(0) = A_0$ .

In other words, a mathematical model can consist of an initial-value problem

# NEWTON'S LAW OF COOLING/WARMING

According to Newton's empirical law of cooling/warming, the rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the temperature of the surrounding medium, the so-called ambient temperature.

If  $T(t)$  represents the temperature of a body at time  $t$ ,  $T_m$  the temperature of the surrounding medium, and  $dT/dt$  the rate at which the temperature of the body changes, then Newton's law of cooling/warming translates into the mathematical statement

$$\frac{dT}{dt} = k(T - T_m) \quad (3)$$

where  $k$  is a constant of proportionality.

**SPREAD OF A DISEASE** A contagious disease—for example, a flu virus—is spread throughout a community by people coming into contact with other people. Let  $x(t)$  denote the number of people who have contracted the disease and  $y(t)$  denote the number of people who have not yet been exposed. It seems reasonable to assume that the rate  $dx/dt$  at which the disease spreads is proportional to the number of encounters, or *interactions*, between these two groups of people. If we assume that the number of interactions is jointly proportional to  $x(t)$  and  $y(t)$ —that is, proportional to the product  $xy$ —then

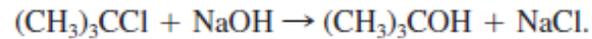
$$\frac{dx}{dt} = kxy, \quad (4)$$

where  $k$  is the usual constant of proportionality. Suppose a small community has a fixed population of  $n$  people. If one infected person is introduced into this community, then it could be argued that  $x(t)$  and  $y(t)$  are related by  $x + y = n + 1$ . Using this last equation to eliminate  $y$  in (4) gives us the model

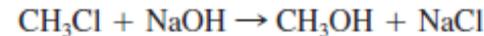
$$\frac{dx}{dt} = kx(n + 1 - x). \quad (5)$$

An obvious initial condition accompanying equation (5) is  $x(0) = 1$ .

**CHEMICAL REACTIONS** The disintegration of a radioactive substance, governed by the differential equation (1), is said to be a **first-order reaction**. In chemistry a few reactions follow this same empirical law: If the molecules of substance  $A$  decompose into smaller molecules, it is a natural assumption that the rate at which this decomposition takes place is proportional to the amount of the first substance that has not undergone conversion; that is, if  $X(t)$  is the amount of substance  $A$  remaining at any time, then  $dX/dt = kX$ , where  $k$  is a negative constant since  $X$  is decreasing. An example of a first-order chemical reaction is the conversion of  $t$ -butyl chloride,  $(\text{CH}_3)_3\text{CCl}$ , into  $t$ -butyl alcohol,  $(\text{CH}_3)_3\text{COH}$ :



Only the concentration of the  $t$ -butyl chloride controls the rate of reaction. But in the reaction



one molecule of sodium hydroxide,  $\text{NaOH}$ , is consumed for every molecule of methyl chloride,  $\text{CH}_3\text{Cl}$ , thus forming one molecule of methyl alcohol,  $\text{CH}_3\text{OH}$ , and one molecule of sodium chloride,  $\text{NaCl}$ . In this case the rate at which the reaction proceeds is proportional to the product of the remaining concentrations of  $\text{CH}_3\text{Cl}$  and  $\text{NaOH}$ . To describe this second reaction in general, let us suppose *one* molecule of a substance  $A$  combines with *one* molecule of a substance  $B$  to form *one* molecule of a substance  $C$ . If  $X$  denotes the amount of chemical  $C$  formed at time  $t$  and if  $\alpha$  and  $\beta$  are, in turn, the amounts of the two chemicals  $A$  and  $B$  at  $t = 0$  (the initial amounts), then the instantaneous amounts of  $A$  and  $B$  not converted to chemical  $C$  are  $\alpha - X$  and  $\beta - X$ , respectively. Hence the rate of formation of  $C$  is given by

$$\frac{dX}{dt} = k(\alpha - X)(\beta - X), \quad (6)$$

where  $k$  is a constant of proportionality. A reaction whose model is equation (6) is said to be a **second-order reaction**.

# MIXTURES

The mixing of two salt solutions of differing concentrations gives rise to a first-order differential equation for the amount of salt contained in the mixture.

Let us suppose that a large mixing tank initially holds 300 gallons of brine (that is, water in which a certain number of pounds of salt has been dissolved).

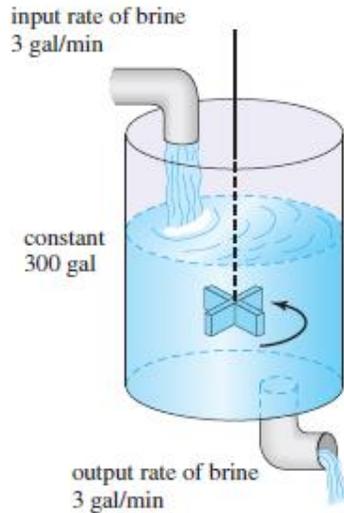


FIGURE 1.3.1 Mixing tank

Another brine solution is pumped into the large tank at a rate of 3 gallons per minute; the concentration of the salt in this inflow is 2 pounds per gallon. When the solution in the tank is well stirred, it is pumped out at the same rate as the entering solution. See Figure 1.3.1. If  $A(t)$  denotes the amount of salt (measured in pounds) in the tank at time  $t$ , then the rate at which  $A(t)$  changes is a net rate:

$$\frac{dA}{dt} = \left( \begin{array}{c} \text{input rate} \\ \text{of salt} \end{array} \right) - \left( \begin{array}{c} \text{output rate} \\ \text{of salt} \end{array} \right) = R_{in} - R_{out} \quad (7)$$

The input rate  $R_{in}$  at which salt enters the tank is the product of the inflow concentration of salt and the inflow rate of fluid. Note that  $R_{in}$  is measured in pounds per minute:

$$R_{in} = \begin{array}{ccc} \text{concentration} & & \text{input rate} \\ \text{of salt} & & \text{of brine} \\ \text{in inflow} & & \\ \downarrow & & \downarrow \\ (2 \text{ lb/gal}) & \cdot & (3 \text{ gal/min}) = (6 \text{ lb/min}). \end{array}$$

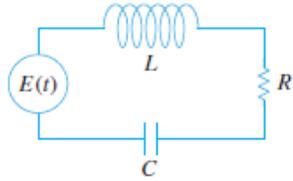
Now, since the solution is being pumped out of the tank at the same rate that it is pumped in, the number of gallons of brine in the tank at time  $t$  is a constant 300 gallons. Hence the concentration of the salt in the tank as well as in the outflow is  $c(t) = A(t)/300$  lb/gal, so the output rate  $R_{out}$  of salt is

$$R_{out} = \begin{array}{ccc} \text{concentration} & & \text{output rate} \\ \text{of salt} & & \text{of brine} \\ \text{in outflow} & & \\ \downarrow & & \downarrow \\ \left( \frac{A(t)}{300} \text{ lb/gal} \right) & \cdot & (3 \text{ gal/min}) = \frac{A(t)}{100} \text{ lb/min}. \end{array}$$

The net rate (7) then becomes

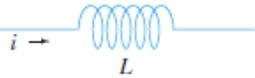
$$\frac{dA}{dt} = 6 - \frac{A}{100} \quad \text{or} \quad \frac{dA}{dt} + \frac{1}{100}A = 6. \quad (8)$$



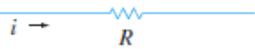


(a) LRC-series circuit

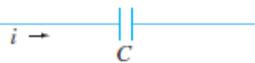
*Inductor*  
inductance  $L$ : henries (h)  
voltage drop across:  $L \frac{di}{dt}$



*Resistor*  
resistance  $R$ : ohms ( $\Omega$ )  
voltage drop across:  $iR$



*Capacitor*  
capacitance  $C$ : farads (f)  
voltage drop across:  $\frac{1}{C} q$



(b)

**FIGURE 1.3.3** Symbols, units, and voltages. Current  $i(t)$  and charge  $q(t)$  are measured in amperes (A) and coulombs (C), respectively

**SERIES CIRCUITS** Consider the single-loop series circuit shown in Figure 1.3.3(a), containing an inductor, resistor, and capacitor. The current in a circuit after a switch is closed is denoted by  $i(t)$ ; the charge on a capacitor at time  $t$  is denoted by  $q(t)$ . The letters  $L$ ,  $R$ , and  $C$  are known as inductance, resistance, and capacitance, respectively, and are generally constants. Now according to Kirchhoff's second law, the impressed voltage  $E(t)$  on a closed loop must equal the sum of the voltage drops in the loop. Figure 1.3.3(b) shows the symbols and the formulas for the respective voltage drops across an inductor, a capacitor, and a resistor. Since current  $i(t)$  is related to charge  $q(t)$  on the capacitor by  $i = dq/dt$ , adding the three voltages

$$L \frac{di}{dt} = L \frac{d^2q}{dt^2}, \quad iR = R \frac{dq}{dt}, \quad \text{and} \quad \frac{1}{C} q$$

and equating the sum to the impressed voltage yields a second-order differential equation

$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = E(t). \quad (11)$$

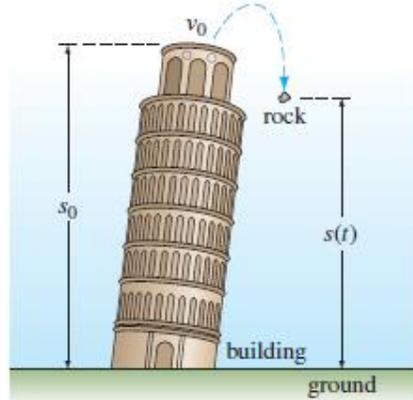


FIGURE 1.3.4 Position of rock measured from ground level

**FALLING BODIES** To construct a mathematical model of the motion of a body moving in a force field, one often starts with Newton's second law of motion. Recall from elementary physics that **Newton's first law of motion** states that a body either will remain at rest or will continue to move with a constant velocity unless acted on by an external force. In each case this is equivalent to saying that when the sum of the forces  $F = \sum F_k$ —that is, the *net* or resultant force—acting on the body is zero, then the acceleration  $a$  of the body is zero. **Newton's second law of motion** indicates that when the net force acting on a body is not zero, then the net force is proportional to its acceleration  $a$  or, more precisely,  $F = ma$ , where  $m$  is the mass of the body.

Now suppose a rock is tossed upward from the roof of a building as illustrated in Figure 1.3.4. What is the position  $s(t)$  of the rock relative to the ground at time  $t$ ? The acceleration of the rock is the second derivative  $d^2s/dt^2$ . If we assume that the upward direction is positive and that no force acts on the rock other than the force of gravity, then Newton's second law gives

$$m \frac{d^2s}{dt^2} = -mg \quad \text{or} \quad \frac{d^2s}{dt^2} = -g. \quad (12)$$

In other words, the net force is simply the weight  $F = F_1 = -W$  of the rock near the surface of the Earth. Recall that the magnitude of the weight is  $W = mg$ , where  $m$  is

the mass of the body and  $g$  is the acceleration due to gravity. The minus sign in (12) is used because the weight of the rock is a force directed downward, which is opposite to the positive direction. If the height of the building is  $s_0$  and the initial velocity of the rock is  $v_0$ , then  $s$  is determined from the second-order initial-value problem

$$\frac{d^2s}{dt^2} = -g, \quad s(0) = s_0, \quad s'(0) = v_0. \quad (13)$$

Although we have not been stressing solutions of the equations we have constructed, note that (13) can be solved by integrating the constant  $-g$  twice with respect to  $t$ . The initial conditions determine the two constants of integration. From elementary physics you might recognize the solution of (13) as the formula  $s(t) = -\frac{1}{2}gt^2 + v_0t + s_0$ .

There are three different types of approaches to, or analyses of, differential equations.

Over the centuries differential equations would often spring from the efforts of a scientist or engineer to describe some physical phenomenon or to translate an empirical or experimental law into mathematical terms. As a consequence a scientist, engineer, or mathematician would often spend **many years of his or her life** trying to find the solutions of a DE.

With a solution in hand, the study of its properties then followed. This quest for solutions is called by some **the analytical approach** to differential equations.

Once they realized that explicit solutions are at best difficult to obtain and at worst impossible to obtain, mathematicians learned that a differential equation itself could be a font of valuable information. It is possible, in some instances, to glean directly from the differential equation answers to questions such as

- › Does the DE actually have solutions?
- › If a solution of the DE exists and satisfies an initial condition, is it the only such solution?
- › What are some of the properties of the unknown solutions?
- › What can we say about the geometry of the solution curves?

Such an approach is **qualitative analysis**.

Finally, if a differential equation cannot be solved by analytical methods, yet we can prove that a solution exists, the next logical query is **Can we somehow approximate the values of an unknown solution?**

Here we enter the realm of **numerical analysis**. An affirmative answer to the last question stems from the fact that a differential equation can be used as a cornerstone for constructing very accurate approximation algorithms.

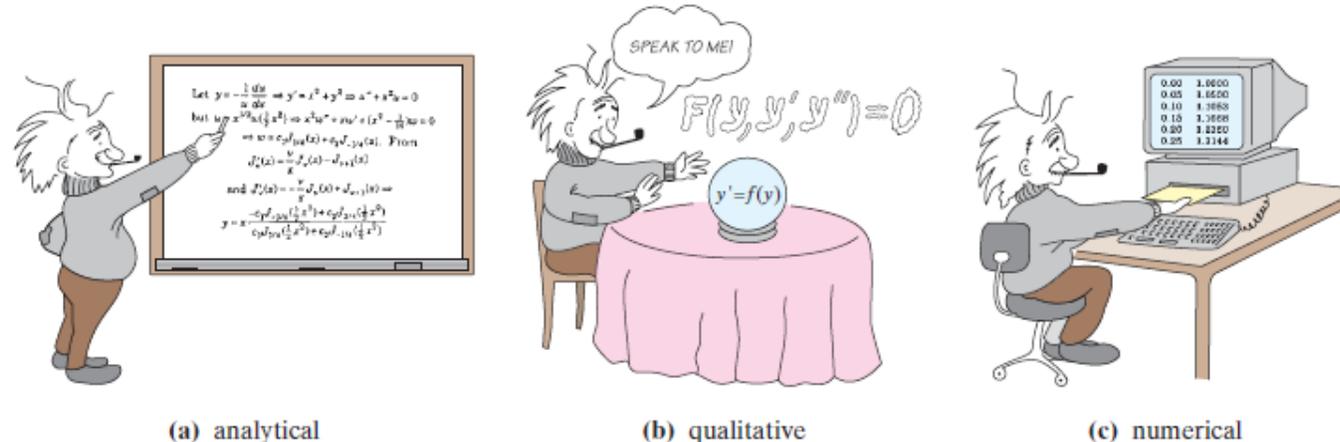


FIGURE 1.3.8 Different approaches to the study of differential equations