7. UNDETERMINED COEFFICIENTS

INTRODUCTION To solve a nonhomogeneous linear differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x), \qquad (1)$$

we must do two things:

- find the complementary function y_c and
- find *any* particular solution y_p of the nonhomogeneous equation (1).

As was discussed In Section 5, the general solution of (1) is $y = y_c + y_p$. The complementary function y_c is the general solution of the associated homogeneous DE of (1), that is,

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

In Section 6, we saw how to solve these kinds of equations when the coefficients were constants. Our goal in the present section is to develop a method for obtaining **particular solutions.**

Now, let us consider the following nonhomogeneous differential equation with constant coefficients:

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = F(x),$$

Let's give some definitions.

Definition

We shall call a function a UC function *if it is* either *(1) a function defined by one of the following:*

(*i*) x^n, *where n is a positive integer or zero,*

(*ii*) e^{ax}, *where a is a constant* $\neq 0$,

(*iii*) $\sin(bx + c)$, *where b and c are constants,* $b \neq 0$,

(*iv*) $\cos(bx + c)$, *where b and c are constants,* $b \neq 0$,

or (2) *a function defined as a finite product of two or more functions of these four types.*

Example

Examples of UC functions of the four basic types (*i*), (*ii*), (*iii*), (*iv*) of the preceding definition are those defined, respectively, by

$$x^3, \quad e^{-2x}, \quad \sin(3x/2), \quad \cos(2x + \pi/4).$$

Examples of UC functions defined as finite products of two or more of these four basic types are those defined, respectively, by

$$x^2e^{3x}, \quad x\cos 2x, \quad e^{5x}\sin 3x,$$

$$\sin 2x \cos 3x, \quad x^3e^{4x}\sin 5x.$$

The method of undetermined coefficients applies when the nonhomogeneous function F in the differential equation is a finite linear combination of UC functions. Observe that given a UC function f, each successive derivative of f is either itself a constant multiple of a UC function or else a linear combination of UC functions.

Definition

Consider a UC function f. The set of functions consisting of f itself and all linearly independent UC functions of which the successive derivatives of f are either constant multiples or linear combinations will be called the UC set *of f.*

Example

The function f defined for all real x by $f(x) = x^3$ is a UC function. Computing derivatives of f, we find

$$f'(x) = 3x^2, \qquad f''(x) = 6x, \qquad f'''(x) = 6 = 6 \cdot 1, \qquad f^{(n)}(x) = 0 \quad \text{for} \quad n > 3.$$

The linearly independent UC functions of which the successive derivatives of f are either constant multiples or linear combinations are those given by

$$x^2, \qquad x, \qquad 1.$$

Thus the *UC set* of x^3 is the set $S = \{x^3, x^2, x, 1\}$.

Example

The function f defined for all real x by $f(x) = \sin 2x$ is a UC function. Computing derivatives of f, we find

$$f'(x) = 2\cos 2x, \qquad f''(x) = -4\sin 2x, \qquad \cdots.$$

The only linearly independent UC function of which the successive derivatives of f are constant multiples or linear combinations is that given by $\cos 2x$. Thus the *UC set* of $\sin 2x$ is the set $S = \{\sin 2x, \cos 2x\}$.

Table of UC functions

	UC function	UC set
1	x^n	$\{x^n, x^{n-1}, x^{n-2}, \ldots, x, 1\}$
2	e^{ax}	$\{e^{ax}\}$
3	$\sin(bx+c)$ or $\cos(bx+c)$	$\{\sin(bx+c), \cos(bx+c)\}$
4	$x^n e^{ax}$	$\{x^n e^{ax}, x^{n-1}e^{ax}, x^{n-2}e^{ax}, \ldots, xe^{ax}, e^{ax}\}$
5	$x^n \sin(bx+c)$ or $x^n \cos(bx+c)$	$\{x^n \sin(bx+c), x^n \cos(bx+c), x^{n-1}\sin(bx+c), x^{n-1}\cos(bx+c), \ldots, x\sin(bx+c), x\cos(bx+c), \sin(bx+c), \cos(bx+c)\}$
6	$e^{ax}\sin(bx+c)$ or $e^{ax}\cos(bx+c)$	$\{e^{ax}\sin(bx+c), e^{ax}\cos(bx+c)\}$
7	$x^n e^{ax}\sin(bx+c)$ or $x^n e^{ax}\cos(bx+c)$	$\{x^n e^{ax}\sin(bx+c), x^n e^{ax}\cos(bx+c), x^{n-1}e^{ax}\sin(bx+c), x^{n-1}e^{ax}\cos(bx+c), \ldots, xe^{ax}\sin(bx+c), xe^{ax}\cos(bx+c), e^{ax}\sin(bx+c), e^{ax}\cos(bx+c)\}$

Example

The function f defined for all real x by $f(x) = x^2 \sin x$ is the product of the two UC functions defined by x^2 and $\sin x$. Hence f is itself a UC function. Computing derivatives of f, we find

$$f'(x) = 2x \sin x + x^2 \cos x,$$

$$f''(x) = 2 \sin x + 4x \cos x - x^2 \sin x,$$

$$f'''(x) = 6 \cos x - 6x \sin x - x^2 \cos x, \qquad \cdots.$$

No "new" types of functions will occur from further differentiation. Each derivative of f is a linear combination of certain of the six UC functions given by $x^2 \sin x$, $x^2 \cos x$, $x \sin x$, $x \cos x$, $\sin x$, and $\cos x$. Thus the set

$$S = \{x^2 \sin x, x^2 \cos x, x \sin x, x \cos x, \sin x, \cos x\}$$

is the *UC set* of $x^2 \sin x$. Note carefully that x^2, x, and 1 are *not* members of this UC set.

The Method Step by Step

- **Step 1:** Solve the homogeneous equation and write the set of independent solution’s set, i.e. Fundamental Set of Solutions (FSS).
- **Step 2:** Find UC set (S) of the right hand side function ($F(x)$).
- **Step 3:** If the set S includes one or more member of FSS, then multiply each member of S by the lowest positive integer power of x. So, the new set , S_1, does not include any member of FSS.
- **Step 4:** The linear combination of S_1, is in the form of particular solution

SOLVE QUESTIONS

CAUTION

Keep in mind that the method of undetermined coefficients applies only to nonhomogeneities that are polynomials, exponentials, sines or cosines, or products of these functions.