8. VARIATION OF PARAMETERS

We have seen that the method of undetermined coefficients is a simple procedure for determining a particular solution when the equation has

- constant coefficients and
- the nonhomogeneous term is of a special type.

Here we present a more general method, called **variation of parameters,** for finding a particular solution.

Consider the nonhomogeneous linear second-order equation

(1) $$ay'' + by' + cy = f(t)$$

and let $\{y_1(t), y_2(t)\}$ be two linearly independent solutions for the corresponding homogeneous equation

$$ay'' + by' + cy = 0 \ .$$

Then we know that a general solution to this homogeneous equation is given by

(2) $$y_h(t) = c_1y_1(t) + c_2y_2(t) \ ,$$

where c_1 and c_2 are constants. To find a particular solution to the nonhomogeneous equation, the strategy of variation of parameters is to replace the constants in (2) by functions of t. That is, we seek a solution of (1) of the form[†]

(3) $$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t) \ .$$

Because we have introduced two unknown functions, $v_1(t)$ and $v_2(t)$, it is reasonable to expect that we can impose two equations (requirements) on these functions. Naturally, one of these equations should come from (1). Let's therefore plug $y_p(t)$ given by (3) into (1). To accomplish this, we must first compute $y_p'(t)$ and $y_p''(t)$. From (3) we obtain

$$y_p' = (v_1'y_1 + v_2'y_2) + (v_1 y_1' + v_2y_2') \ .$$

To simplify the computation and to avoid second-order derivatives for the unknowns v_1, v_2 in the expression for y_p'', we impose the requirement

(4) $$v_1'y_1 + v_2'y_2 = 0 \ .$$

Thus, the formula for y_p' becomes

(5) $$y_p' = v_1y_1' + v_2y_2' \ ,$$

and so

(6) $$y_p'' = v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2'' \ .$$

Now, substituting y_p, y_p', and y_p'', as given in (3), (5), and (6), into (1), we find

(7) $$\begin{aligned} f &= ay_p'' + by_p' + cy_p \\ &= a(v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2'') + b(v_1y_1' + v_2y_2') + c(v_1y_1 + v_2y_2) \\ &= a(v_1'y_1' + v_2'y_2') + v_1(ay_1'' + by_1' + cy_1) + v_2(ay_2'' + by_2' + cy_2) \\ &= a(v_1'y_1' + v_2'y_2') + 0 + 0 \end{aligned}$$

since y_1 and y_2 are solutions to the homogeneous equation. Thus, (7) reduces to

(8) $$v_1'y_1' + v_2'y_2' = \frac{f}{a} \ .$$

To summarize, if we can find v_1 and v_2 that satisfy both (4) and (8), that is,

(9) $$\begin{aligned} y_1v_1' + y_2v_2' &= 0 \ , \\ y_1'v_1' + y_2'v_2' &= \frac{f}{a} \ , \end{aligned}$$

then y_p given by (3) will be a particular solution to (1). To determine v_1 and v_2, we first solve the linear system (9) for v_1' and v_2'. Algebraic manipulation or Cramer's rule (see Appendix D) immediately gives

$$v_1'(t) = \frac{-f(t)y_2(t)}{a[y_1(t)y_2'(t) - y_1'(t)y_2(t)]} \quad \text{and} \quad v_2'(t) = \frac{f(t)y_1(t)}{a[y_1(t)y_2'(t) - y_1'(t)y_2(t)]} \ ,$$

where the bracketed expression in the denominator (the Wronskian) is never zero because y_1 *and* y_2 are linearly independent. Upon integrating these equations, we finally obtain

$$(10) \qquad v_1(t) = \int \frac{-f(t)y_2(t)}{a[y_1(t)y_2'(t) - y_1'(t)y_2(t)]}dt \quad \text{and} \quad v_2(t) = \int \frac{f(t)y_1(t)}{a[y_1(t)y_2'(t) - y_1'(t)y_2(t)]}dt \ .$$

Let's review this procedure.

Method of Variation of Parameters

To determine a particular solution to $ay'' + by' + cy = f$:

(a) Find two linearly independent solutions $\{y_1(t), y_2(t)\}$ to the corresponding homogeneous equation and take

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t) \ .$$

(b) Determine $v_1(t)$ and $v_2(t)$ by solving the system in (9) for $v_1'(t)$ and $v_2'(t)$ and integrating.

(c) Substitute $v_1(t)$ and $v_2(t)$ into the expression for $y_p(t)$ to obtain a particular solution.

Of course, in step (b) one could use the formulas in (10), but and are so easy to derive that you are advised not to memorize them.

Appendix D (From Linear Algebra

D CRAMER'S RULE

When a system of n linear equations in n unknowns has a unique solution, determinants can be used to obtain a formula for the unknowns. This procedure is called **Cramer's rule.** When n is small, these formulas provide a simple procedure for solving the system.

Suppose that for a system of n linear equations in n unknowns,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \ , \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \ , \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \ , \end{aligned} \tag{1}$$

the coefficient matrix

$$\mathbf{A} := \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{2}$$

has a nonzero determinant. Then Cramer's rule gives the solutions

$$x_i = \frac{\det \mathbf{A}_i}{\det \mathbf{A}} , \qquad i = 1, 2, \ldots, n , \tag{3}$$

where $\mathbf{A}_i$ is the matrix obtained from $\mathbf{A}$ by replacing the ith column of $\mathbf{A}$ by the column vector

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

consisting of the constants on the right-hand side of system (1).

Example 1 Use Cramer's rule to solve the system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \ , \\ 2x_1 + x_2 + x_3 &= 9 \ , \\ x_1 - x_2 - 2x_3 &= 1 \ . \end{aligned}$$

Solution We first compute the determinant of the coefficient matrix:

$$\det \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix} = 12 \ .$$

Using formula (3), we find

$$x_1 = \frac{1}{12} \det \begin{bmatrix} 0 & 2 & -1 \\ 9 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix} = \frac{48}{12} = 4 \ ,$$

$$x_2 = \frac{1}{12} \det \begin{bmatrix} 1 & 0 & -1 \\ 2 & 9 & 1 \\ 1 & 1 & -2 \end{bmatrix} = \frac{-12}{12} = -1 \ ,$$

$$x_3 = \frac{1}{12} \det \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 9 \\ 1 & -1 & 1 \end{bmatrix} = \frac{24}{12} = 2 \ . \ \blacklozenge$$

SOLVE QUESTIONS

9. THE CAUCHY-EULER EQUATION

INTRODUCTION

- The same relative ease with which we were able to find explicit solutions of higher-order linear differential equations with constant coefficients in the preceding sections does not, in general, carry over to linear equations with variable coefficients.
- However, the type of differential equation that we consider in this section is an exception to this rule; it is a linear equation with variable coefficients whose general solution can always be expressed in terms of powers of x, sines, cosines, and logarithmic functions.
- Moreover, its method of solution is quite similar to that for constant-coefficient equations in that an auxiliary equation must be solved.

CAUCHY-EULER EQUATION A linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x),$$

where the coefficients $a_n, a_{n-1}, \ldots, a_0$ are constants, is known as a **Cauchy-Euler equation.** The observable characteristic of this type of equation is that the degree $k = n, n-1, \ldots, 1, 0$ of the monomial coefficients x^k matches the order k of differentiation $d^k y/dx^k$:

$$a_n x^{\overset{\text{same}}{\downarrow}n} \frac{d^{\downarrow n} y}{dx^n} + a_{n-1} x^{\overset{\text{same}}{\downarrow}n-1} \frac{d^{\downarrow n-1} y}{dx^{n-1}} + \cdots .$$

Theorem

The transformation $x = e^t$ *reduces the equation*

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_{n-1} x y' + a_n y = F(x)$$

to a linear differential equation with constant coefficients.

We shall give the proof of this theorem for the second order case!!

SOLVE QUESTIONS