4. MODELLING WITH FIRST ORDER DIFFERENTIAL EQUATIONS

In this section, we solve some of the linear first-order models that were introduced in Section 3.

4.1 GROWTH AND DECAY PROBLEMS

The initial-value problem

$$\frac{dx}{dt} = kx, \qquad x(t_0) = x_0 \qquad (1)$$

where k is a constant of proportionality, serves as a model for diverse phenomena involving either growth or decay.

We saw in previous section that in biological applications the rate of growth of certain populations (bacteria, small animals) over short periods of time is proportional to the population present at time *t.*

- Knowing the population at some arbitrary initial time t_0, we can then use the solution of (1) to predict the population in the future —that is, at times $t > t_0$.
- The constant of proportionality k in (1) can be determined from the solution of the initial-value problem, using a subsequent measurement of x at a time $t_1 > t_0$.
- In physics and chemistry (1) is seen in the form of *a first-order reaction* —that is, a reaction whose rate, or velocity, $\frac{dx}{dt}$ is directly proportional to the amount x of a substance that is unconverted or remaining at time t.
- The decomposition, or decay, of U-238 (uranium) by radioactivity into Th-234 (thorium) is a first-order reaction.

EXAMPLE 1 Bacterial Growth

A culture initially has P_0 number of bacteria. At $t = 1$ h the number of bacteria is measured to be $\frac{3}{2}P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t, determine the time necessary for the number of bacteria to triple.

EXAMPLE 2 Half-Life of Plutonium

A breeder reactor converts relatively stable uranium 238 into the isotope plutonium 239. After 15 years it is determined that 0.043% of the initial amount A_0 of plutonium has disintegrated. Find the half-life of this isotope if the rate of disintegration is proportional to the amount remaining.

EXAMPLE 3

Bacteria grow in a culture at a rate proportional to the amount present. Initially, 300 strands of the bacteria are in the culture and after two hours that number has grown by 20 percent. Find (a) an expression for the approximate number of strands in the culture at any time t and (b) the time needed for the bacteria to double its initial size.

4.2 NEWTON’S LAW OF COOLING/WARMING

In equation (3) of Section 3 we saw that the mathematical formulation of Newton’s empirical law of cooling/warming of an object is given by the linear first-order differential equation

$$\frac{dT}{dt} = k(T - T_m), \qquad (2)$$

where k is a constant of proportionality, $T(t)$ is the temperature of the object for $t > 0$, and T_m is the ambient temperature—that is, the temperature of the medium around the object.

EXAMPLE 4 Cooling of a Cake

When a cake is removed from an oven, its temperature is measured at 300° F. Three minutes later its temperature is 200° F. How long will it take for the cake to cool off to a room temperature of 70° F?

EXAMPLE 5

A metal bar at a temperature of 100° F is placed in a room at a constant temperature of 0°F. If after 20 minutes the temperature of the bar is 50° F, find (a) the time it will take the bar to reach a temperature of 25° F and (b) the temperature of the bar after 10 minutes.

4.3 FALLING BODY PROBLEMS

Consider a vertically falling body of mass m that is being influenced only by gravity g and an air resistance that is proportional to the velocity of the body. Assume that both gravity and mass remain constant and, for convenience, choose the downward direction as the positive direction

Newton's second law of motion*: The net force acting on a body is equal to the time rate of change of the momentum of the body; or, for constant mass*

$$F = m\frac{dv}{dt} \qquad (3)$$

where F is the net force on the body and v is the velocity of the body, both at time t.

For the problem at hand, there are two forces acting on the body: (1) the force due to gravity given by the weight w of the body, which equals mg, and (2) the force due to air resistance given by $-kv$, where $k > 0$ is a constant of proportionality.

The minus sign is required because this force opposes the velocity; that is, it acts in the upward, or negative, direction. The net force F on the body is, therefore, $F = mg - kv$.

Substituting this result into (3), we obtain

$$mg - kv = m\frac{dv}{dt}$$

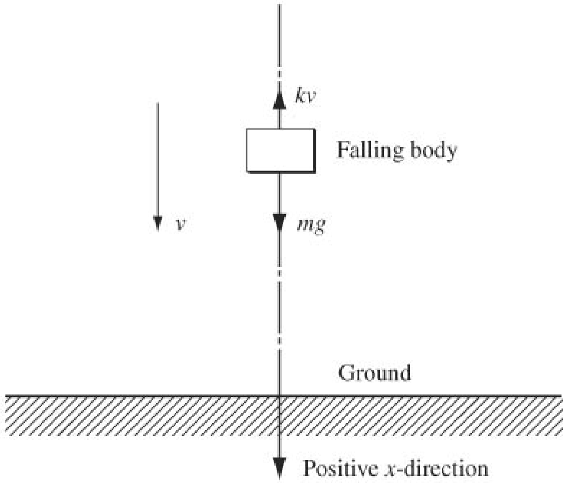
Or

$$\frac{dv}{dt} + \frac{k}{m}v = g \qquad (4)$$

as the equation of motion for the body.

If air resistance is negligible or nonexistent, then $k = 0$ and (4) simplifies to

$$\frac{dv}{dt} = g \qquad (g \cong 32)$$



EXAMPLE 6

A body is dropped from a height of 300ft with an initial velocity of 30ft/sec. Assuming no air resistance, find (a) an expression for the velocity of the body at any time t and (b) the time required for the body to hit the ground.

EXAMPLE 7

A body of mass 2 slugs is dropped from a height of 450 ft with an initial velocity of 10 ft/sec. Assuming no air resistance, find (a) an expression for the velocity of the body at any time t and (b) the time required for the body to hit the ground.

4.4 ORTHOGONAL TRAJECTORIES

Consider a one-parameter family of curves in the xy-plane defined by

$$F(x, y, c) = 0 \qquad (5)$$

where c denotes the parameter. The problem is to find another one-parameter family of curves, called the **orthogonal trajectories** of the family (5) and given analytically by

$$G(x, y, k) = 0 \qquad (6)$$

such that every curve in this new family (6) intersects at right angles every curve in the original family (5).

We first implicitly differentiate (5) with respect to x, then eliminate c between this derived equation and (5). This gives an equation connecting $x, y, and\ y'$, which we solve for y' to obtain a differential equation of the form

$$\frac{dy}{dx} = f(x, y)$$

The orthogonal trajectories of (5) are the solutions of

$$\frac{dy}{dx} = \frac{-1}{f(x, y)}$$

EXAMPLE 8

Find the orthogonal trajectories of the following family of curves

$a)\quad y = cx^2$

$b)\quad y = \frac{cx}{1+x}$

$c)\quad x^2 + 3y^2 = cy$

Supplementary Problems

1) The population of a certain state is known to grow at a rate proportional to the number of people presently living in the state. If after 10 years the population has trebled and if after 20 years the population is 150,000, find the number of people initially living in the state.

2) A bar of iron, previously heated to 1200° C, is cooled in a large bath of water maintained at a constant temperature of 50° C. The bar cools by 200° in the first minute. How much longer will it take to cool a second 200°?

3) A body weighing 160 Ib is dropped 2000 ft above ground with no initial velocity. As it falls, the body encounters a force due to air resistance proportional to its velocity. If the limiting velocity of this body is 320 ft/sec, find (a) an expression for the velocity of the body at any time t and (b) an expression for the position of the body at any time t.

4) Find the orthogonal trajectories of the family of curves $y = ce^x$.