11.3. INVERSE TRANSFORMS

THE INVERSE PROBLEM If $F(s)$ represents the Laplace transform of a function $f(t)$, that is, $\mathscr{L}\{f(t)\} = F(s)$, we then say $f(t)$ is the **inverse Laplace transform** of $F(s)$ and write $f(t) = \mathscr{L}^{-1}\{F(s)\}$.

Transform	Inverse Transform
$\mathscr{L}\{1\} = \dfrac{1}{s}$	$1 = \mathscr{L}^{-1}\left\{\dfrac{1}{s}\right\}$
$\mathscr{L}\{t\} = \dfrac{1}{s^2}$	$t = \mathscr{L}^{-1}\left\{\dfrac{1}{s^2}\right\}$
$\mathscr{L}\{e^{-3t}\} = \dfrac{1}{s+3}$	$e^{-3t} = \mathscr{L}^{-1}\left\{\dfrac{1}{s+3}\right\}$

We shall see shortly that in the application of the Laplace transform to equations we are not able to determine an unknown function $f(t)$ directly; rather, we are able to solve for the Laplace transform $F(s)$ of $f(t)$; but from that knowledge we ascertain f by computing $f(t) = \mathscr{L}^{-1}\{F(s)\}$. The idea is simply this: Suppose $F(s) = \dfrac{-2s + 6}{s^2 + 4}$ is a Laplace transform; find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$.

Some Inverse Transforms

$$\text{(a)}\quad 1 = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$\text{(b)}\quad t^n = \mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\},\quad n = 1, 2, 3, \ldots \qquad \text{(c)}\quad e^{at} = \mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\}$$

$$\text{(d)}\quad \sin kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2 + k^2}\right\} \qquad \text{(e)}\quad \cos kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2 + k^2}\right\}$$

$$\text{(f)}\quad \sinh kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2 - k^2}\right\} \qquad \text{(g)}\quad \cosh kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2 - k^2}\right\}$$

In evaluating inverse transforms, it often happens that a function of s under consideration does not match exactly the form of a Laplace transform $F(s)$ given in a table. So, we use some tools:

1. It may be necessary to **"fix up"** the function of s by multiplying and dividing by an appropriate constant.
2. We can use partial fractions and perfect square.

Inverse transform is also linear

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha \mathscr{L}^{-1}\{F(s)\} + \beta \mathscr{L}^{-1}\{G(s)\},$$

EXAMPLE Termwise Division and Linearity

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{-2s+6}{s^2+4}\right\}$.

SOLUTION We first rewrite the given function of s as two expressions by means of termwise division

$$\mathscr{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\} \overset{\text{termwise division}}{=} \mathscr{L}^{-1}\left\{\frac{-2s}{s^2+4}+\frac{6}{s^2+4}\right\} \overset{\text{linearity and fixing up constants}}{=} -2\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{6}{2}\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= -2\cos 2t + 3\sin 2t.$$

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Partial Fractions

We briefly review this method. Recall from calculus that a rational function of the form $P(s)/Q(s)$, where $P(s)$ and $Q(s)$ are polynomials with the degree of P less than the degree of Q, has a partial fraction expansion whose form is based on the linear and quadratic factors of $Q(s)$. (We assume the coefficients of the polynomials to be real numbers.) There are three cases to consider:

1. Nonrepeated linear factors.
2. Repeated linear factors.

1. Nonrepeated Linear Factors

If $Q(s)$ can be factored into a product of distinct linear factors,

$$Q(s) = (s - r_1)(s - r_2)\cdots(s - r_n) ,$$

where the r_i's are all distinct real numbers, then the partial fraction expansion has the form

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s - r_1} + \frac{A_2}{s - r_2} + \cdots + \frac{A_n}{s - r_n} ,$$

where the A_i's are real numbers. There are various ways of determining the constants $A_1, \ldots, A_n$. In the next example, we demonstrate two such methods.

2. Repeated Linear Factors

Let $s - r$ be a factor of $Q(s)$ and suppose $(s - r)^m$ is the highest power of $s - r$ that divides $Q(s)$. Then the portion of the partial fraction expansion of $P(s)/Q(s)$ that corresponds to the term $(s - r)^m$ is

$$\frac{A_1}{s - r} + \frac{A_2}{(s - r)^2} + \cdots + \frac{A_m}{(s - r)^m} ,$$

where the A_i's are real numbers.

Examples

Determine $\mathscr{L}^{-1}\{F\}$, where

$$F(s) = \frac{7s - 1}{(s + 1)(s + 2)(s - 3)} .$$

Determine $\mathscr{L}^{-1}\left\{\frac{s^2 + 9s + 2}{(s - 1)^2(s + 3)}\right\} .$

Now let us give example about perfect square

› Find the inverse transform of the function

$$F(s) = \frac{s+4}{s^2+4s+8}$$

Convolution

CONVOLUTION If functions f and g are piecewise continuous on the interval $[0, \infty)$, then a special product, denoted by $f * g$, is defined by the integral

$$f * g = \int_0^t f(\tau)\, g(t - \tau)\, d\tau$$

and is called the **convolution** of f and g. The convolution $f * g$ is a function of t.

Example: Find the convolution of $f(x) = x \; and \; g(x) = e^x$.

Theorem (Convolution Theorem)

If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\,\mathscr{L}\{g(t)\} = F(s)G(s).$$

11.4. SOLVING INITIAL VALUE PROBLEMS

- Our goal is to show how Laplace transforms can be used to solve initial value problems for linear differential equations. Recall that we have already studied ways of solving such initial value problems in previous sections.
- These previous methods required that we first find a *general solution* of the differential equation and then use the initial conditions to determine the desired solution.
- As we will see, the method of Laplace transforms leads to the solution of the initial value problem **without first finding a general solution.**

Method of Laplace Transforms

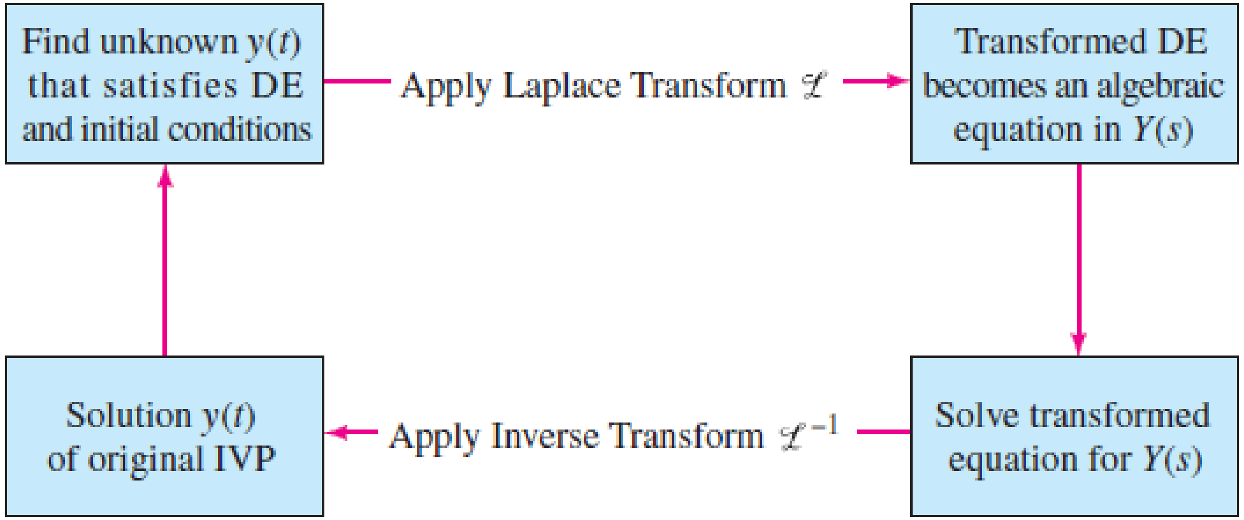
To solve an initial value problem:

(a) Take the Laplace transform of both sides of the equation.

(b) Use the properties of the Laplace transform and the initial conditions to obtain an equation for the Laplace transform of the solution and then solve this equation for the transform.

(c) Determine the inverse Laplace transform of the solution by looking it up in a table or by using a suitable method (such as partial fractions) in combination with the table.

The procedure also can be given by following diagram



The method will be explained in detailed in the class and several examples will be solved.