MTH 102 Calculus -II

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Topic-7-Polar Coordinates

4.1 Polar Coordinates

A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. Usually we use Cartesian coordinates, which are directed distances from two perpendicular axes. Here we describe a coordinate system introduced by Newton, called the polar coordinate system, which is more convenient for many purposes.

We choose a point in the plane that is called the **pole** (or origin) and is labeled O. Then we draw a ray (half-line) starting at O called the **polar axis.** This axis is usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.
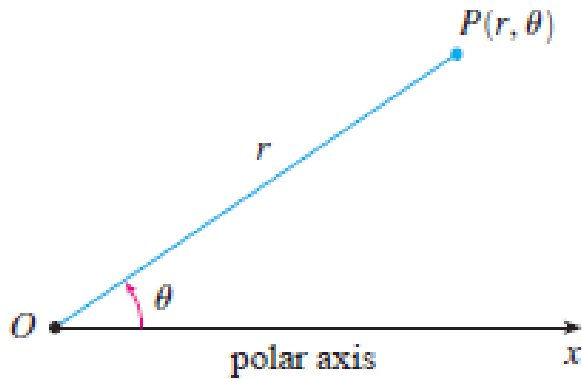
$P(r, \theta)$

r

θ

O

polar axis

x

FIGURE 1

(r, θ)

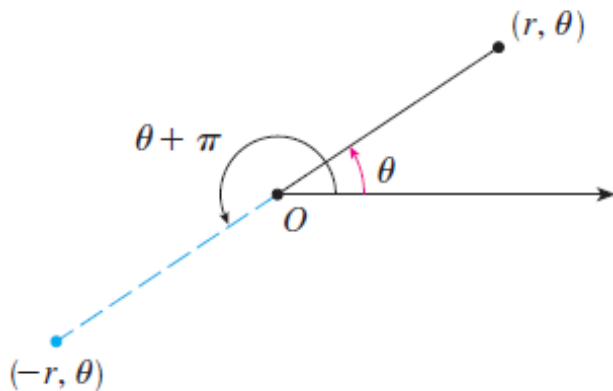$\theta + \pi$

θ

O

$(-r, \theta)$

FIGURE 2

If P is any other point in the plane, let r be the distance from O to P and let θ be the angle between the polar axis and the line OP as in Figure 1. Then the point P is represented by the ordered pair (r, θ) and r, θ are called **polar coordinates** of P. We use the convention that an angle r is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If $P = O$, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of θ.

We extend the meaning of polar coordinates (r, θ) to the case in which r is negative by agreeing that, as in Figure 2, the points $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance $|r|$ from O, but on opposite sides of O.

If , $r > 0$ the point (r, θ) lies in the same quadrant as θ; if $r < 0$, it lies in the quadrant on the opposite side of the pole.

Notice that $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.

Example 1 Plot the points whose polar coordinates are given.

$$(a) \left(1, \frac{5\pi}{4}\right), (b) \ (2, 3\pi), (c) \left(2, -\frac{2\pi}{3}\right), (d) \left(-3, \frac{3\pi}{4}\right)$$

Solution

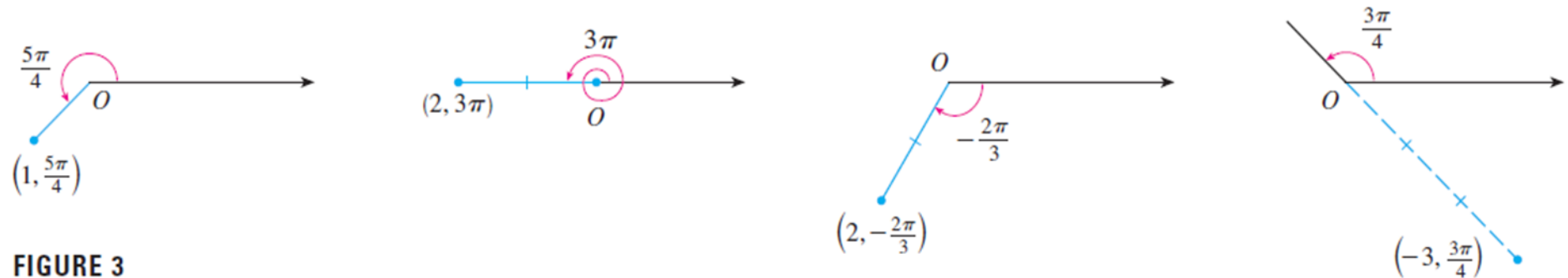The points are plotted in Figure 3.

FIGURE 3

In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations. For instance, the point $\left(1, \frac{5\pi}{4}\right)$ in Example 1(a) can be written as $\left(1, \frac{-3\pi}{4}\right)$ or $\left(1, \frac{13\pi}{4}\right)$ or $\left(-1, \frac{\pi}{4}\right)$
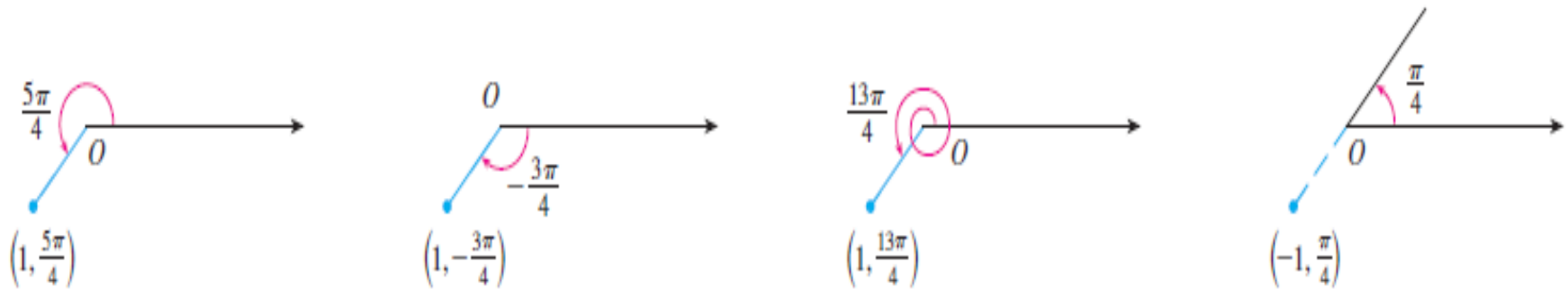

FIGURE 4

In fact, since a complete counterclockwise rotation is given by an angle 2π, the point represented by polar coordinates (r, θ) is also represented by

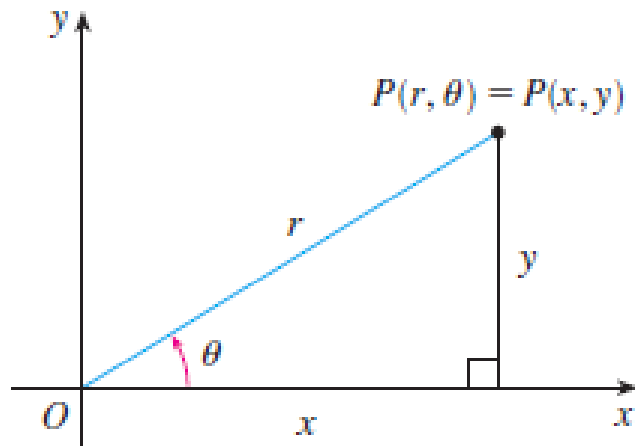$$(r, \theta + 2n\pi) \text{ and } (-r, \theta + (2n+1)\pi)$$

where n is any integer.

The connection between polar and Cartesian coordinates can be seen from Figure 5, in which the pole corresponds to the origin and the polar axis coincides with the positive x-axis. If the point has Cartesian coordinates (x, y) and polar coordinates (r, θ), then, from the figure, we have

$$x = r\cos\theta, \ y = r\sin\theta \ \dots(1)$$


$P(r, \theta) = P(x, y)$

FIGURE 5

Equations 1 allow us to find the Cartesian coordinates of a point when the polar coordinates are known. To find r and θ when x and y are known, we use the equations

$$r^2 = x^2 + y^2 \ , \tan\theta = \frac{y}{x} \dots (2)$$

Polar Curves

The graph of a polar equation $r = f(\theta)$, or more generally , $F(r, \theta)$ consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.
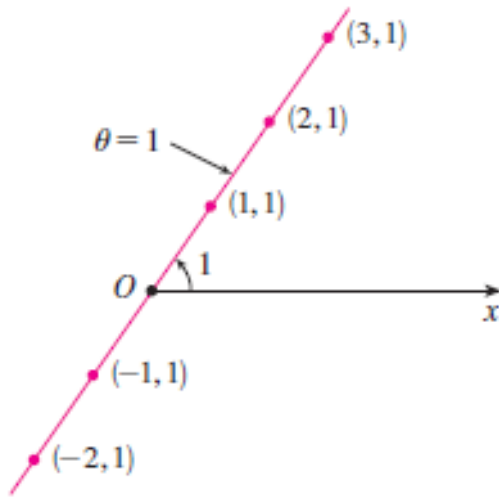
Example: What curve is represented by the polar equation $r = 2$?
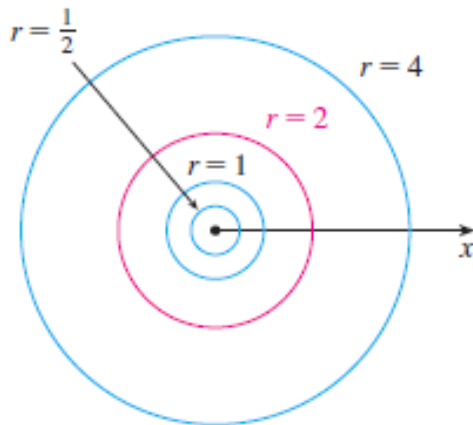
The curve consists of all points (r, θ) with $r = 2$. Since r represents the distance from the point to the pole, the curve $r = 2$ represents the circle with center O and radius 2. In general, the equation $r = a$ represents a circle with center O and radius $|a|$.

Example: Sketch the polar curve $\theta = 1$.

This curve consists of all points (r,θ) such that the polar angle θ is 1 radian. It is the straight line that passes through O and makes an angle of 1 radian with the polar axis as in the Figure. Notice that the points $(r,1)$ on the line with $r > 0$ are in the first quadrant, whereas those with $r < 0$ are in the third quadrant.

Example: Sketch the polar curve r $= 2cos\theta$.

θ	$r = 2\cos\theta$
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2
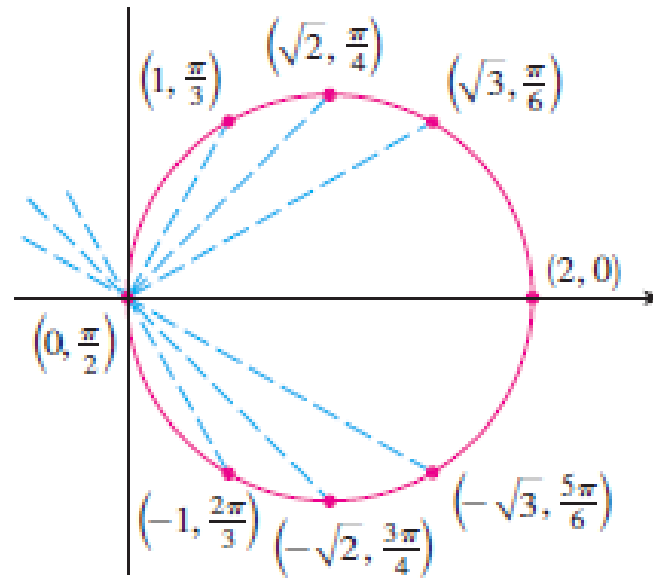
We find the values of r for some convenient values of θ and plot the corresponding points (r,θ). Then we join these points to sketch the curve, which appears to be a circle. We have used only values of θ between 0 and π , since if we let θ increase beyond π, we obtain the same points again.

Areas and Lengths in Polar Coordinates

In this section we develop the formula for the area of a region whose boundary is given by a polar equation. We need to use the formula for the area of a sector of a circle:
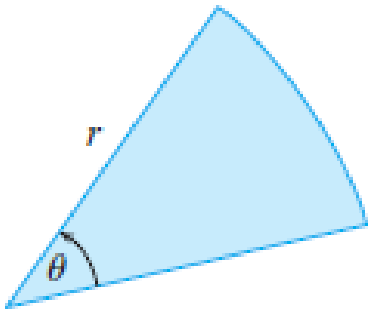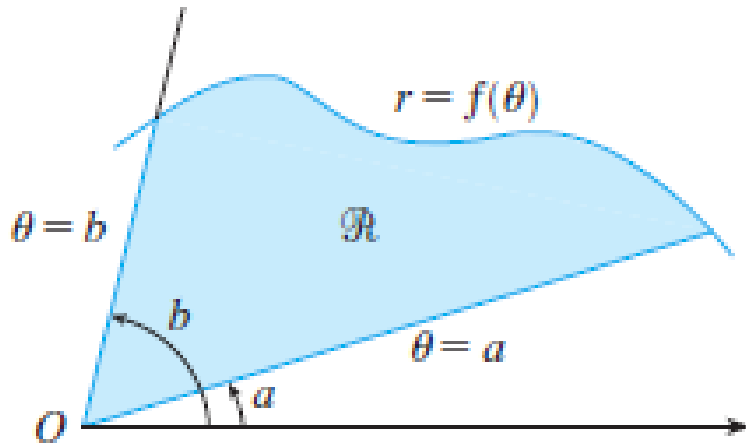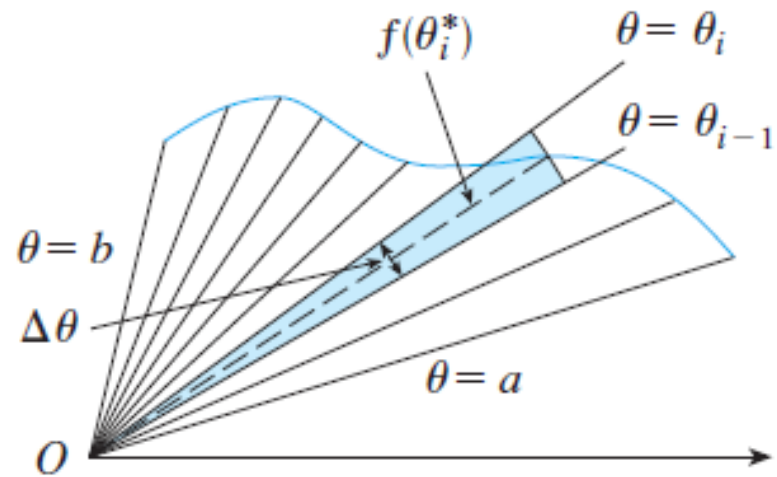
$$A = \frac{1}{2}r^2\theta \dots (1)$$

where, as in Fig 1 r is the radius and θ is the radian measure of the central angle.

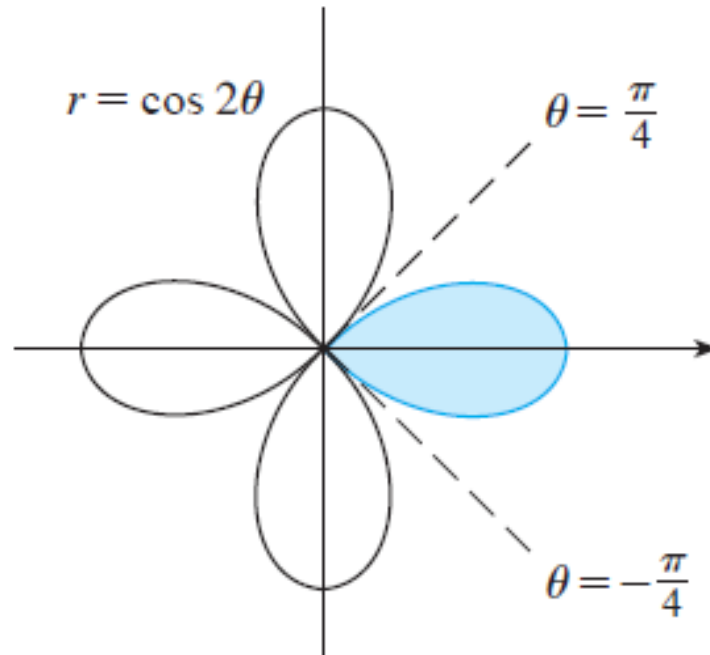**FIGURE 1**

$r = f(\theta)$

$\theta = b$

$\mathcal{R}$

b

$\theta = a$

O

a

FIGURE 2

$f(\theta_i^*)$

$\theta = \theta_i$

$\theta = \theta_{i-1}$

$\theta = b$

$\Delta\theta$

$\theta = a$

O

FIGURE 3

Example-1 Find the area enclosed by one loop of the four-leaved rose $r = cos2\theta$.

$r = \cos 2\theta$ $\theta = \dfrac{\pi}{4}$

$\theta = -\dfrac{\pi}{4}$

FIGURE 4