

MTH 102 Calculus -II

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Topic-8-Infinite Sequences



Infinite Sequences

A sequence can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, \dots, a_n, \dots$$

The number a_1 is called **the first term**, a_2 is **the second term**, and in general a_n is **the n th term**. We will deal exclusively with infinite sequences and so each term a_n will have a successor a_{n+1} .

Notice that for every positive integer n there is a corresponding number a_n and so a sequence can be defined as a function whose domain is the set of positive integers. But we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n . The sequence $\{a_1, a_2, a_3, \dots\}$ is also denoted by (a_n) , $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

Now let's solve some examples!!!



Examples

Some sequences can be defined by giving a formula for the n th term. In the following examples we give three descriptions of the sequence: one by using the preceding notation, another by using the defining formula, and a third by writing out the terms of the sequence. Notice that n doesn't have to start at 1.

$$(a) \left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty} \quad a_n = \frac{n}{n+1} \quad \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n}{n+1}, \dots \right\}$$

$$(b) \left\{ \frac{(-1)^n(n+1)}{3^n} \right\} \quad a_n = \frac{(-1)^n(n+1)}{3^n} \quad \left\{ -\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \dots, \frac{(-1)^n(n+1)}{3^n}, \dots \right\}$$

$$(c) \left\{ \sqrt{n-3} \right\}_{n=3}^{\infty} \quad a_n = \sqrt{n-3}, \quad n \geq 3 \quad \{0, 1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n-3}, \dots\}$$

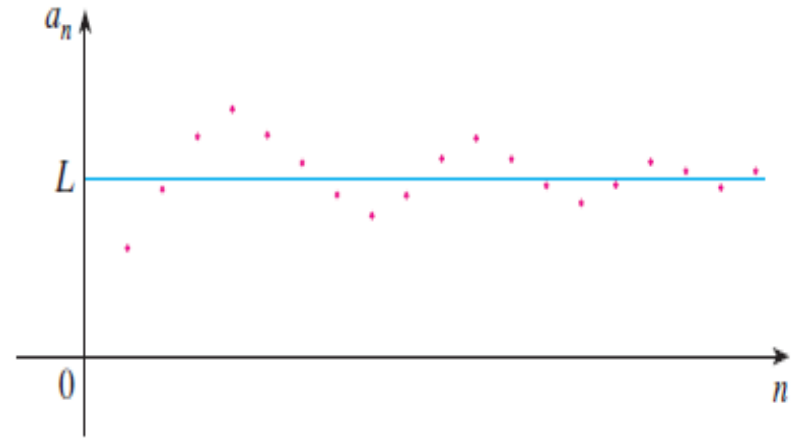
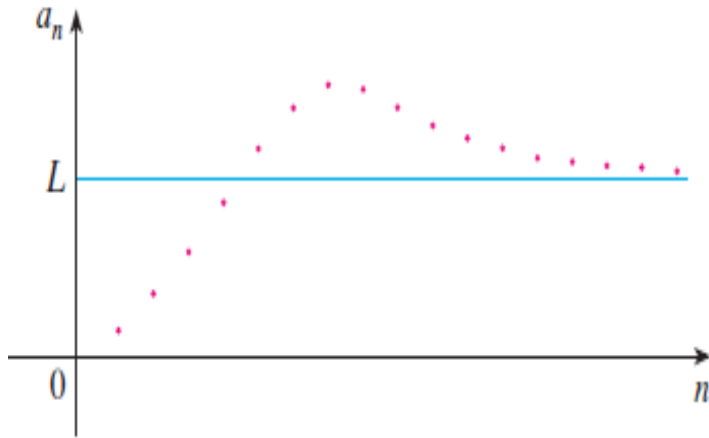
$$(d) \left\{ \cos \frac{n\pi}{6} \right\}_{n=0}^{\infty} \quad a_n = \cos \frac{n\pi}{6}, \quad n \geq 0 \quad \left\{ 1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \dots, \cos \frac{n\pi}{6}, \dots \right\}$$

1 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{or} \quad a_n \rightarrow L \text{ as } n \rightarrow \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n \rightarrow \infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

The following figure illustrates Definition 1 by showing the graphs of two sequences that have the limit L .



A more precise version of Definition 1 is as follows.

2 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{or} \quad a_n \rightarrow L \text{ as } n \rightarrow \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if } n > N \quad \text{then} \quad |a_n - L| < \varepsilon$$

If $\lim_{n \rightarrow \infty} a_n = \infty$, then the sequence is divergent but in a special way. We say that $\{a_n\}$ diverges to ∞ .

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} ca_n = c \lim_{n \rightarrow \infty} a_n$$

$$\lim_{n \rightarrow \infty} c = c$$

$$\lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} \quad \text{if } \lim_{n \rightarrow \infty} b_n \neq 0$$

$$\lim_{n \rightarrow \infty} a_n^p = \left[\lim_{n \rightarrow \infty} a_n \right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

The Squeeze Theorem can also be adapted for sequences as follows

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$.

6 Theorem

If $\lim_{n \rightarrow \infty} |a_n| = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

10 Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \dots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is **monotonic** if it is either increasing or decreasing.

11 Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M \quad \text{for all } n \geq 1$$

It is **bounded below** if there is a number m such that

$$m \leq a_n \quad \text{for all } n \geq 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.