Lecture 5: Real Vector Spaces

Elif Tan

Ankara University

Real Vector Spaces

- A vector in the plane is a 2×1 matrix (2-vector)

$$x = \begin{bmatrix} x \\ y \end{bmatrix}; x, y \in \mathbb{R}.$$

- A vector in the space is a 3×1 matrix (3-vector)

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; x, y, z \in \mathbb{R}.$$

- We also represent a vector in the plane as a directed line segment for physical applications. In $\mathbb{R}^2$, for each vector $\begin{bmatrix} x \\ y \end{bmatrix}$, there is a corresponding point (x, y), and for each point (x, y), there is a unique vector $\begin{bmatrix} x \\ y \end{bmatrix}$. In algebraically, all these representations behave in a same manner.

Real Vector Spaces

Definition (Real Vector Space)

A real vector space is a set of V of elements on which have two operations $\oplus$ and $\odot$ satisfy the following properties:

- $\oplus : \begin{array}{l} V \times V \longrightarrow V \\ (u,v) \longrightarrow u \oplus v \end{array}$ $\forall u, v \in V, u \oplus v \in V$.

    1. $\forall u, v \in V, u \oplus v = v \oplus u$
    2. $\forall u, v, w \in V, u \oplus (v \oplus w) = (u \oplus v) \oplus w$
    3. For any $u \in V, \exists 0 \in V; u \oplus 0 = 0 \oplus u = u$
    4. For each $u \in V, \exists -u \in V; u \oplus -u = -u \oplus u = 0.$

Definition

- $\odot : \begin{array}{ccc} \mathbb{R} \times V & \longrightarrow & V \\ (c, u) & \longrightarrow & c \odot u \end{array} \quad \forall u \in V, \forall c \in \mathbb{R}, c \odot u \in V.$

  1. $\forall u, v \in V, \forall c \in \mathbb{R}, c \odot (u \oplus v) = c \odot u \oplus c \odot v$
  2. $\forall u \in V, \forall c, d \in \mathbb{R}, (c + d) \odot u = c \odot u \oplus d \odot u$
  3. $\forall u \in V, \forall c, d \in \mathbb{R}, c \odot (d \odot u) = (c.d) \odot u$
  4. $\forall u \in V, 1 \in \mathbb{R}, 1 \odot u = u.$

- We denote $(V, \oplus, \odot)$ is a real vector space.
- The elements of $(V, \oplus, \odot)$ are called as vectors.
- The elements of $\mathbb{R}$ are called as scalars.
- The operations $\oplus$ and $\odot$ are called as vector additon and scalar multilication, respectively.
- Note that the inverse of a vector is unique.

Examples

$(\mathbb{R}^n, \oplus, \odot)$ is a real vector space.
$(\mathbb{R}, +, .)$ is a real vector space.
$(\mathbb{R}^+, +, .)$ is not a real vector space, because the identity element of the operation $+$ doesn't exist.

Theorem

Let $(V, \oplus, \odot)$ *be a real vector space. For* $u \in V$ *and* $c \in \mathbb{R}$,

(i) $0 \odot u = 0$

(ii) $c \odot 0 = 0$

(iii) $c \odot u = 0 \Rightarrow c = 0 \vee u = 0$

(iv) $(-1) \odot u = -u$.

Subspaces

Definition (Subspace)

Let $(V, \oplus, \odot)$ be a real vector space and $\emptyset \neq W \subset V$. If W is a real vector space with the operations in V, then W is called a subspace of V $(W < V)$.

To verify that a subset W of a vector sace V is a subspace, it is enough to check the following conditions.

Theorem

Let $(V, \oplus, \odot)$ be a real vector space and $\emptyset \neq W \subset V$. Then

$$W < V \iff \begin{array}{ll} (i) & \forall u, v \in W, u \oplus v \in W \\ (ii) & \forall u \in W, \forall c \in \mathbb{R}, c \odot u \in W. \end{array}$$

Subspaces

Example

$V := \mathbb{R}^3$ is a real vector space with the standart operations $\oplus$ and $\odot$.

$$W_1 := \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; x + z = 0 \right\} < V.$$

$$W_2 := \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; x + z = 7 \right\} \nless V.$$