Lecture 8: Linear Transformations

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Linear Transformations

Definition (Coordinate)

Let $S = \{v_1, v_2, ..., v_n\}$ be an ordered basis for the n-dimensional vector space $(V, \oplus, \odot)$. Then every vector v in V can be uniquely expressed in the form

$$v = a_1 \odot v_1 \oplus a_2 \odot v_2 \oplus ... \oplus a_n \odot v_n$$

where $a_1, a_2, ..., a_n$ are scalars. The coordinate vector of v with respect to the ordered basis S is defined by

$$[v]_S := \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

The entries of $[v]_S$ are called the coordinates of v with respect to the basis S. Note that there is a one-to-one correspondence between v and $[v]_S$.

Definition (Transition Matrix)

Let $S = \{v_1, v_2, ..., v_n\}$ and $T = \{w_1, w_2, ..., w_n\}$ be an ordered basis for the n-dimensional vector space $(V, \oplus, \odot)$. The transition matrix from the basis T to S is defined by

$$P_{S \leftarrow T} = [[w_1]_S \, [w_2]_S \ldots [w_n]_S]_{n \times n}$$

and the coordinate vector of v wrt S can be written as

$$[v]_S = P_{S \leftarrow T} \, [v]_T \, .$$

Note that the transition matrix is nonsingular matrix and we have

$$P^{-1}_{S \leftarrow T} = P_{T \leftarrow S}.$$

Example

Consider the ordered basis for $\mathbb{R}^3$

$$S = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and

$$T = \left\{ w_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, w_3 = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \right\}.$$

Find the transition matrix from the basis T to S.

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Solution:

$$\begin{aligned}
w_1 &= a_1 \odot v_1 \oplus a_2 \odot v_2 \oplus a_3 \odot v_3 \Rightarrow [w_1]_S = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\
w_2 &= b_1 \odot v_1 \oplus b_2 \odot v_2 \oplus b_3 \odot v_3 \Rightarrow [w_2]_S = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \\
w_3 &= c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus c_3 \odot v_3 \Rightarrow [w_3]_S = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}
\end{aligned}$$

$$P_{S \leftarrow T} = [[w_1]_S \; [w_2]_S \; [w_3]_S]_{3\times 3} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix}.$$

Linear Transformations

Definition (Linear Transformation)

Let $(V, \oplus, \odot)$ and $(W, \boxplus, \boxdot)$ be real vector spaces. $L : V \longrightarrow W$ is called a linear transformation if the following conditions hols:

$$(i)\, \forall u, v \in V, L(u \oplus v) = L(u) \boxplus L(v)$$

$$(ii)\, \forall u \in V, \forall c \in \mathbb{R}, L(c \odot u) = c \boxdot L(u).$$

Linear Transformations

Definition

A linear transformation $L : V \to W$ is called one-to-one if $L(v_1) = L(v_2)$ implies that $v_1 = v_2$ for $v_1, v_2 \in V$.
A linear transformation $L : V \to W$ is called onto if for each $w \in W, \exists v \in V$ such that $L(v) = w$.

Linear Transformations

Definition

Let $L : V \rightarrow W$ be a linear transformation.
The kernel of L is defined by

$$KerL = \{v \in V \mid L(v) = 0_W\}.$$

The range of L is defined by

$$RangeL = L(V) = \{w \in W \mid \exists v \in V; L(v) = w\}.$$

Theorem

Let $L : V \to W$ be a linear transformation. Then we have the following results:

1. $L(0_V) = 0_W$
2. $KerL < V$
3. L *is one-to-one* $\Leftrightarrow KerL = \{0_V\}$
4. $RangeL < W$
5. L *is onto* $\Leftrightarrow L(V) = W$.

Theorem

Let $L : V \to W$ *be a linear transformation with* $dimV = n$*, then*

$$\dim V = \dim KerL + \dim RangeL.$$

Note that $\dim RangeL$ is called as "rank" of L and $\dim KerL$ is called as "nullity" of L.

Example

Let $L : \mathbb{R}^3 \to \mathbb{R}^2$, $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 + x_2 - 3x_3 \end{bmatrix}$ be a linear transformation. Find the rank of L.

Solution:

$$
\begin{aligned}
KerL &= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = 0_W \right\} \\
&= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 + x_2 - 3x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \\
&= \left\{ \begin{bmatrix} 3x_3 \\ -3x_3 \\ x_3 \end{bmatrix} \mid x_3 \in \mathbb{R} \right\}.
\end{aligned}
$$

Thus $\left\{ \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \right\}$ is a basis for $KerL$ and $\dim KerL = 1$. Since

$$\dim V = \dim KerL + \dim RangeL,$$

then we have

$$3 = 1 + rankL.$$

Therefore $rankL = 2$.

Definition (Isomorphism)

Let $(V, \oplus, \odot)$ and $(W, \boxplus, \boxdot)$ be real vector spaces. L is called an ismorphism if $L : V \longrightarrow W$ is a linear transformation that is one-to-one and onto. We show it as $V \cong W$.

Linear Transformations

Theorem

1. *Let V be an n-dimensional vector space. Then $V \cong \mathbb{R}^n$.*
2. *Let V and W be finite dimensional vector spaces.*
   $V \cong W \Leftrightarrow \dim V = \dim W$.