Lecture 12: Inner Product Spaces

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Definition (Inner Product Space)

Let $(V, \oplus, \odot)$ be a real vector space. If the function $\langle , \rangle : V \times V \to \mathbb{R}$ satisfies the following properties, then V is called an inner product space and the function $\langle , \rangle$ is called an inner product function.

(i) $\forall u \in V, \langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0 \Leftrightarrow u = 0$

(ii) $\forall u, v \in V, \langle u, v \rangle = \langle v, u \rangle$

(iii) $\forall u, v, w \in V, \langle u \oplus v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

(iv) $\forall u, v \in V, \forall c \in \mathbb{R}, \langle c \odot u, v \rangle = c \langle u, v \rangle$.

Inner Product Spaces

Example

For $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$, the standard inner product (dot product) on $\mathbb{R}^n$ is defined by

$$\langle u, v \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Theorem

Let V be an inner product space, and $S = \{u_1, u_2, \ldots, u_n\}$ be an ordered basis for the vector space V. Then the matrix $A = [a_{ij}]_{n\times n}, a_{ij} := \langle a_i, a_j \rangle$ is a symmetric matrix, and for every $u, v \in V$, it determines $\langle u, v \rangle$.

Note that the matrix $A = [a_{ij}]_{n\times n}, a_{ij} = \langle u_i, u_j \rangle$ is called the matrix of the inner product with respect to the ordered basis S.

$$A = \begin{bmatrix} \langle u_1, u_1 \rangle & \langle u_1, u_2 \rangle & \cdots & \langle u_1, u_n \rangle \\ \langle u_2, u_1 \rangle & \langle u_2, u_2 \rangle & \cdots & \langle u_2, u_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_n, u_1 \rangle & \langle u_n, u_2 \rangle & \cdots & \langle u_n, u_n \rangle \end{bmatrix}.$$

Definition (Positive definite matrix)

The $n \times n$ symmetric matrix A is called positive definite matrix if it has the property that

$$\forall 0 \neq x \in \mathbb{R}^n, x^T A x > 0.$$

Theorem

Let $A = [a_{ij}]_{n \times n}$ be a positive definite matrix, and $S = \{u_1, u_2, \ldots, u_n\}$ be an ordered basis for the vector space V. Then the function $\langle , \rangle : V \times V \to \mathbb{R}$ that is defined by $\forall u, v \in V$, $\langle u, v \rangle := [u]_S^T A [v]_S$, is an inner product function on V.

Note that it is not easy to determine when a symmetric matrix is positive definite!

Example

Consider the standard inner product function $\left\langle \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right\rangle = ac + bd$ on $\mathbb{R}^2$. The matrix of the inner product with respect to the ordered basis $S = \left\{ u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is

$$A = \begin{bmatrix} \langle u_1, u_1 \rangle & \langle u_1, u_2 \rangle \\ \langle u_2, u_1 \rangle & \langle u_2, u_2 \rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Conversely, consider the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.($A$ is positive definite matrix,verify it). The inner product with respect to the ordered standard basis in $\mathbb{R}^2$ is

$$\left\langle \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right\rangle = \begin{bmatrix} a & b \end{bmatrix} A \begin{bmatrix} c \\ d \end{bmatrix} = ac + bd.$$

Inner Product Spaces

Definition (Lenght)

Let V be an inner product space.The lenght of $v \in V$ is defined by

$$||v|| = \sqrt{\langle v, v \rangle}.$$

Definition (Distance)

Let V be an inner product space.The distance between u and v in V is defined by

$$d\left(u, v\right) := ||u - v|| = \sqrt{\langle u - v, u - v \rangle}.$$

Theorem (Cauchy-Schwarz Inequality)

Let V be an inner product space. $\forall u, v \in V$,

$$|\langle v, v \rangle| \leq ||u|| \, ||v|| \, .$$

By using Cauchy-Schwarz inequality, we define the cosine of an angle between nonzero vectors u and v in V as

$$\cos\theta := \frac{\langle v, v \rangle}{||u|| \, ||v||}, 0 \leq \theta \leq \pi.$$

Corollary

Let V be an inner product space. Then $\forall u, v \in V$ and $\forall c \in \mathbb{R}$,

1. $||c \odot v|| \leq |c| \, ||v||$
2. $d(u, v) = 0 \Leftrightarrow u = v$
3. $d(u, v) = d(v, u)$
4. $||u \oplus v|| \leq ||u|| + ||v||$ *(Triangle inequality).*

Definition

Let V be an inner product space. The vectors u and v in V are orthogonal if $\langle v, v\rangle = 0$. That is,

$$u \perp v \Leftrightarrow \langle v, v\rangle = 0.$$

A set of S of vectors in V is called ortogonal if any two distinct vectors in S are orthogonal. Additionally, if each vector in S is a unit vector $(||u|| = 1, u \in V)$, then S is called orthonormal.

Theorem

1. *Zero vector is orthogonal with every vector in an inner product space* V*, that is,* $0 \perp v, \forall v \in V$.
2. *If* $S = \{u_1, u_2, \ldots, u_n\}$ *is an orthogonal set of nonzero vectors in an inner product space* V*, then* S *is linearly independent.*