Models with Ordinary Differential Equations

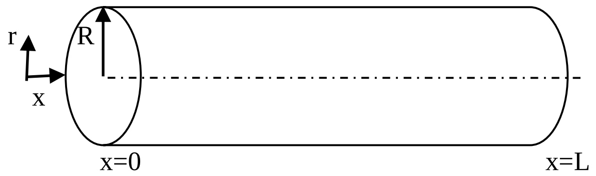
(Momentum balances)

Example:

A Newtonian fluid having a constant density of ρ=912 kg/m^3 is flowing through a horizontal capillary tube. Its inlet pressure is P_0 =1379.5 N/m^2 and its outlet pressure is 100 N/m^2.The tube is smooth and has ID of 0.222mm and a length of 0.1585m.

a) Find the differential equation that gives the fluids radial velocity profile at steady state and indicate the boundary conditions. (R=0 at the center, R=R at the wall)

b) If the Average velocity of the liquid in meters per second is 0.1375 m/s, find the viscosity of the fluid ($V_{max}=2*V_{average\ velocity}$)



Solution :

$$\tau_{rx}A_r\big|_r - \tau_{rx}A_r\big|_{r+\Delta r} + P_0A_x\big|_{x=0} - P_LA_x\big|_{x=L} = 0$$

Steady state; v_x = constant

$$A_r = 2\pi rL, \qquad A_x = 2\pi r\Delta r\ , V = 2\pi rL\Delta r$$

$$\tau_{rx}\,2\pi rL\big|_r - \tau_{rx}\,2\pi rL\big|_{r+\Delta r} + P_0\,2\pi r\Delta r\big|_{x=0} - P_L\,2\pi r\Delta r\big|_{x=L} = 0$$

Divide by $2\pi L\Delta r$;

$$\lim_{\Delta x \to 0}\left(\frac{\tau_{rx} r|_r - \tau_{rx} r|_{r+\Delta r}}{\Delta r}\right) + \frac{(P_0 - P_L) r}{L} = 0$$

$$-\frac{d(\tau_{rx} r)}{dr} + \frac{(P_o - P_L) r}{L} = 0$$

$$\frac{d(\tau_{rx} r)}{dr} = \frac{(P_0 - P_L) r}{L}$$

$$r\frac{d(\tau_{rx})}{dr} + \tau_{rx}\frac{dr}{dr} = \frac{(P_0 - P_L) r}{L}$$

$$r\frac{d(\tau_{rx})}{dr} + \tau_{rx} = \frac{(P_0 - P_L) r}{L}$$

$$\frac{d(\tau_{rx})}{dr} + \frac{\tau_{rx}}{r} = \frac{(P_0 - P_L)}{L}$$ **first order linear ODE.**

$$\frac{d\tau}{dr} + p\tau = Q$$

$$\lambda = e^{-\int P dr} = e^{-\int \frac{1}{r} dr} = e^{-\ln r} = 1/r$$

$$\tau = \frac{1}{r}\left(\frac{P_o - P_l}{L}\int r dr + c\right) = \frac{P_o - P_l}{L}\frac{r^2}{2r} + \frac{c}{r}$$

BK1:r=0(center); Tau=infinity, this is not possible, for this c must be zero

$$\tau_{rx} = -\mu\frac{dv_x}{dr}$$

$$\tau_{rx} = \frac{P_o - P_l}{L}\frac{r}{2} = -\mu\frac{dv_x}{dr}$$

$$dv_x = \frac{-(P_o - P_l)}{L\mu 2} r dr$$

$$v_x = \frac{-(P_o - P_l)}{L\mu 4} r^2 + c2$$

BK2: $r = R \qquad v_x = 0$

$$c2 = \frac{(P_o - P_l)}{L\mu 4} R^2$$

$$v_x = \frac{(P_o - P_l)}{L\mu 4} R^2\left[1 - \left(\frac{r}{R}\right)^2\right]$$

b) At the center, r=0 ,$v_{xmax}=2*v_x$

$$v_{max} = \frac{(P_o - P_l)}{L\mu 4} R^2 \rightarrow 2*0.1375 = \frac{1379.5 - 100}{0.1585*4*\mu}\left(\frac{2.22*10^{-3}}{2}\right)^2 \rightarrow \mu = 0.00904\, kg/ms$$