Models with partial differential equations

(Momentum balances)

Consider an incompressible Newtonian fluid contained between two parallel plates of area A, separated by a distance B as shown in figure 1.The system is initially at rest but at time t=0, the lower plate is set in motion in the z direction at a constant velocity of V_o while the upper plate is kept stationary.

The mathematical expression describing this system as;

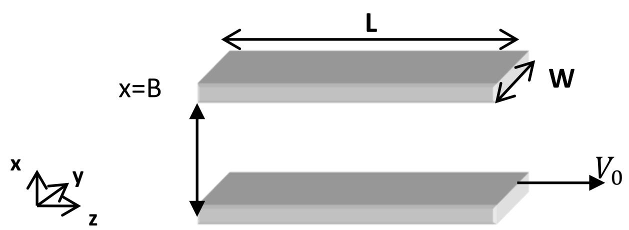

Figure 1.Unsteady flow between parallel plates.

Momentum balance;

$$\tau_{xz}.A_x\big|_x - \tau_{xz}.A_x\big|_{x+\Delta x} + \dot{m}v_z\big|_{L=0} - \dot{m}v_z\big|_{L=L} + P_o.A_z - P_L.A_z = \frac{\partial}{\partial t}(\rho V v_z)$$

$$\dot{m}v_z\big|_{L=0} = \dot{m}v_z\big|_{L=L}$$
$$P_o = P_L$$
$$A_x = L.W$$
$$A_z = \Delta x.W$$
$$V = \Delta x.L.W$$

$$\tau_{xz}.L.W\big|_x - \tau_{xz}.L.W\big|_{x+\Delta x} = \frac{\partial}{\partial t}(\rho.\Delta x.L.W.v_z)$$

Divide by L.W.Δx and taking limΔx→ 0,

$$\lim_{\Delta x \to 0} \frac{\tau_{xz}\big|_x - \tau_{xz}\big|_{x+\Delta x}}{\Delta x} = \frac{\partial}{\partial t}(\rho.v_z)$$

$$-\frac{\partial(\tau_{xz})}{\partial x} = \frac{\partial}{\partial t}(\rho . v_z) \qquad \tau_{xz} = -\mu \frac{\partial v_z}{\partial x}$$

$$\mu \frac{\partial^2 v_z}{\partial x^2} = \rho \frac{\partial v_z}{\partial t}$$

$$\frac{\partial v_z}{\partial t} = \frac{\mu}{\rho} \frac{\partial^2 v_z}{\partial x^2} \quad \ldots\ldots (1)$$ Laminar velocity profile

The initial and boundary conditions

IC: $t = 0$ $v_z = 0$
BC1: $x = 0$ $v_z = V_o$
BC2: $x = B$ $v_z = 0$