

Calculus

Lecture 2

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Limit: Intuitive Meaning

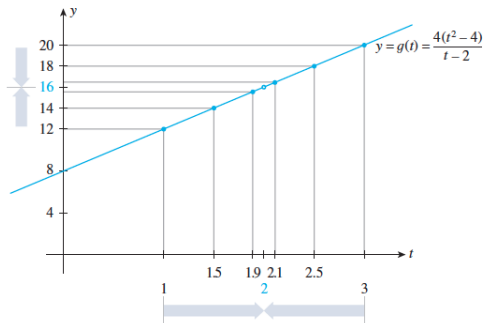
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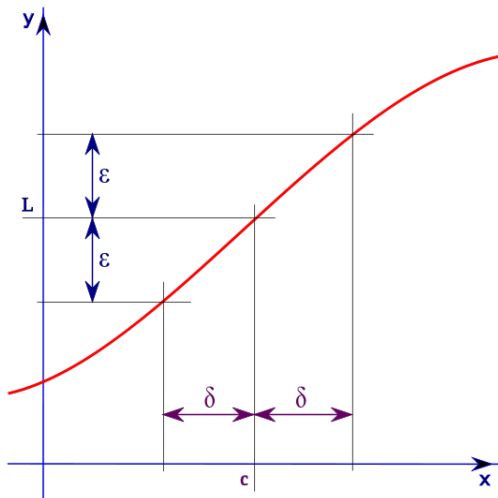
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Symbolically:

$$\lim_{x \rightarrow c} f(x) = L \iff (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in D)(0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon).$$

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Solution

According to the formal definition, the statement is true if and only if confining x to δ units of 5 will inevitably confine $3x - 3$ to ε units of 12. The overall key to showing this implication is to demonstrate how δ and ε must be related to each other such that the implication holds.

Mathematically, we want to show that

$$0 < |x - 5| < \delta \Rightarrow |(3x - 3) - 12| < \varepsilon.$$

Simplifying the equation yields $|x - 5| < \varepsilon/3$. Thus, δ and ε is related by $\delta \leq \varepsilon/3$.

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Theorem

$\lim_{x \rightarrow c^-} f(x) = L = \lim_{x \rightarrow c^+} f(x)$ if and only if $\lim_{x \rightarrow c} f(x) = L$

Heaviside Function: The Heaviside function is often used to specify when something is on or off

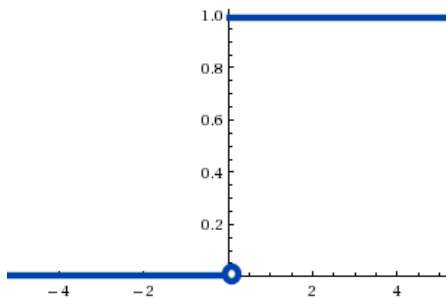
The Heaviside function is defined as

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

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- This function clearly has the limit of 0 for any $x < 0$, and it has the limit of 1 for any $x > 0$.
- Even though this function is defined to be 1 at $x = 0$, it does not have a limit at $x = 0$

An Application: Free-Fall

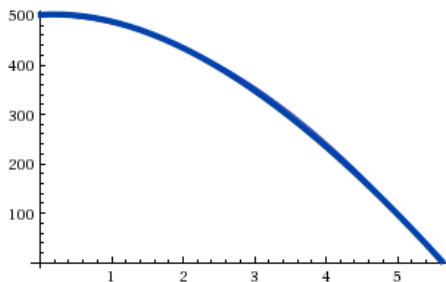
Example

Use the position function $s(t) = -16t^2 + 500$ which gives the height (in feet) of an object that has fallen for seconds from a height of 500 feet. The velocity at time $t = a$ seconds is given by

$$\lim_{t \rightarrow a} \frac{s(a) - s(t)}{a - t}$$

- (a) If a construction worker drops a wrench from a height of 500 feet, how fast will the wrench be falling after 2 seconds?
- (b) If a construction worker drops a wrench from a height of 500 feet, when will the wrench hit the ground? At what velocity will the wrench impact the ground?

Solution



$$(a) \lim_{t \rightarrow 2} \frac{s(2) - s(t)}{2 - t} = -64$$

$$(b) \lim_{t \rightarrow \frac{5\sqrt{5}}{2}} \frac{s\left(\frac{5\sqrt{5}}{2}\right) - s(t)}{\frac{5\sqrt{5}}{2} - t} = -80\sqrt{5}$$