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PHY121 Physics I

Chapter 9 Center of Mass and Linear Momentum

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Chapter 9 Center of Mass and Linear Momentum

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9.1. Center of Mass

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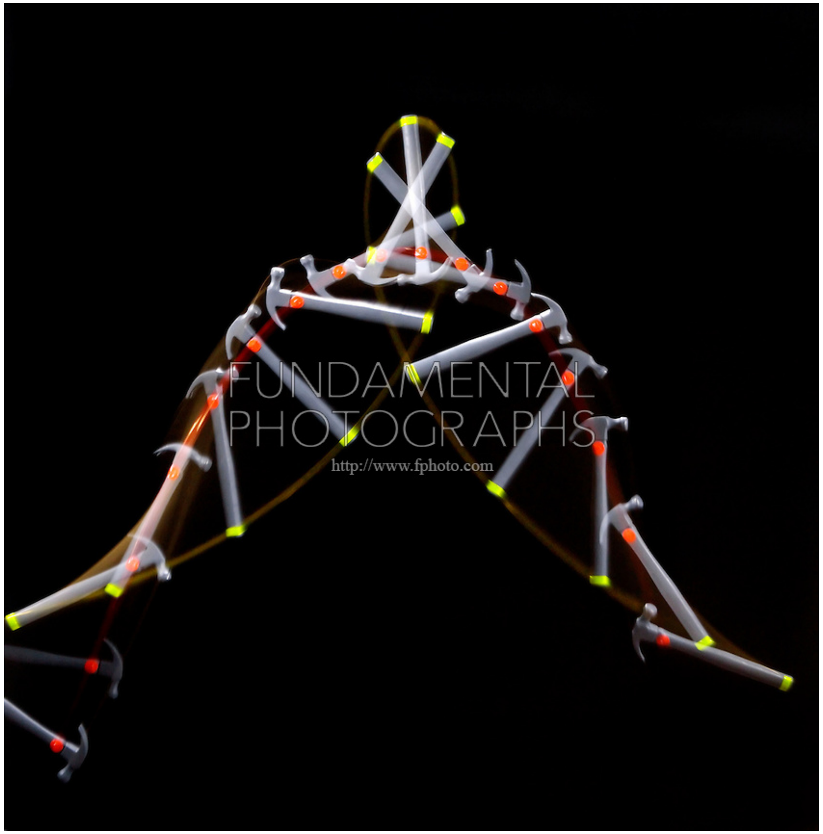
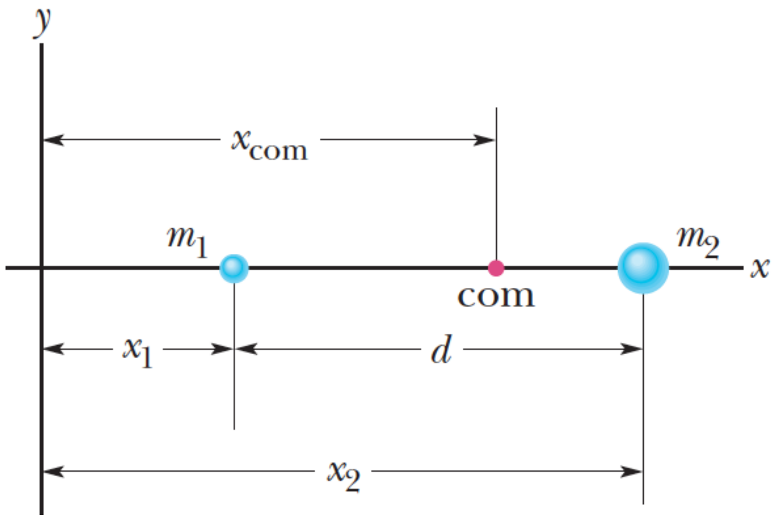


The center of mass of a system of particles is the point that moves as though (1) all of the system's mass were concentrated there and (2) all external forces were applied there.

9.1. Center of Mass

Center of mass for a systems of particles



$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$\boxed{x_{CM} = \frac{m_1 x_1 + m_2 x_2}{M}}$$

İf there is n particles:

$$\boxed{x_{CM} = \frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{M} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i}$$

9.1. Center of Mass

The coordinates of CM for three dimensions

$$\boxed{x_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i}$$

$$\boxed{y_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i}$$

$$\boxed{z_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i}$$

In vector form:

$$\boxed{\vec{r}_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i}$$

9.1. Center of Mass

Solid Bodies

For solid bodies, the coordinates of the center of mass are defined as

$$x_{CM} = \frac{1}{M}\int x\,dm \qquad y_{CM} = \frac{1}{M}\int y\,dm \qquad z_{CM} = \frac{1}{M}\int z\,dm$$

where M is the mass of the object

Uniform objects have uniform density:

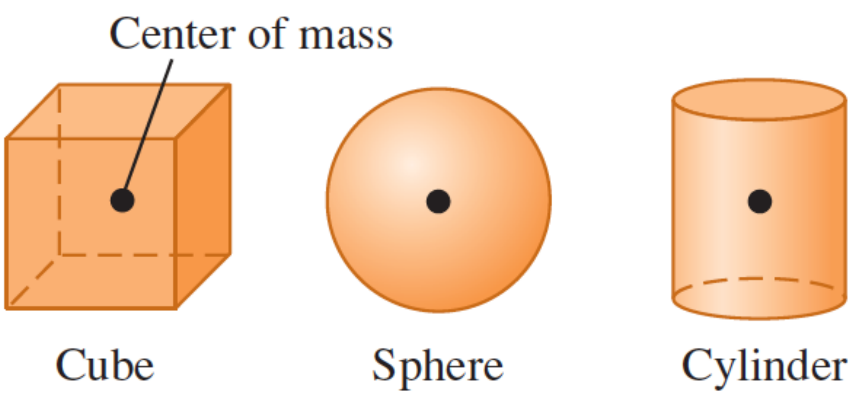
$$\rho = \frac{dm}{dV} = \frac{M}{V}$$

9.1. Center of Mass

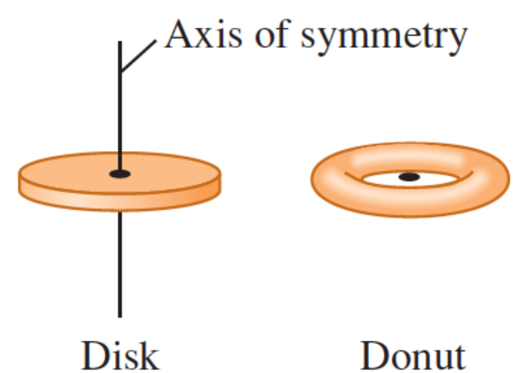
$$x_{KM} = \frac{1}{V}\int x\,dV$$

$$y_{KM} = \frac{1}{V}\int y\,dV$$

$$z_{KM} = \frac{1}{V}\int z\,dV$$



If a homogeneous object has a geometric center, that is where the center of mass is located.



If an object has an axis of symmetry, the center of mass lies along it. As in the case of the donut, the center of mass may not be within the object.

9.2. Newton's 2. Law for a System of Particles

The position vector of CM for a system with n particles:

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_n\vec{r}_n}{M}$$

The first derivation of this equation gives:

$$M\frac{d\vec{r}_{CM}}{dt} = m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + \ldots + m_n\frac{d\vec{r}_n}{dt}$$

$$M\vec{v}_{CM} = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots + m_n\vec{v}_n$$

and the first derivation of this equation gives:

$$M\frac{d\vec{v}_{KM}}{dt} = m_1\frac{d\vec{v}_{\mathbf{1}}}{dt} + m_2\frac{d\vec{v}_2}{dt} + \ldots + m_n\frac{d\vec{v}_n}{dt}$$

$$M\vec{a}_{KM} = m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots + m_n\vec{a}_n$$

9.2. Newton's 2. Law for a System of Particles

Thus, we obtained

$$M\vec{a}_{CM} = m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots + m_n\vec{a}_n$$

$$M\vec{a}_{CM} = \vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_n$$

$$\boxed{M\vec{a}_{CM} = \vec{F}_{net}}$$

- **F_{net} is the net force of all external forces that act on the system. Forces on one part of the system from another part of the system (internal forces) are not included in this equation**

- **M is the total mass of the system and M remains constant during the movement (System is closed)**

- **a_{CM} is the acceleration of the center of mass of the system. This equation gives no information about the acceleration of any other point of the system.**

9.3. Linear Momentum

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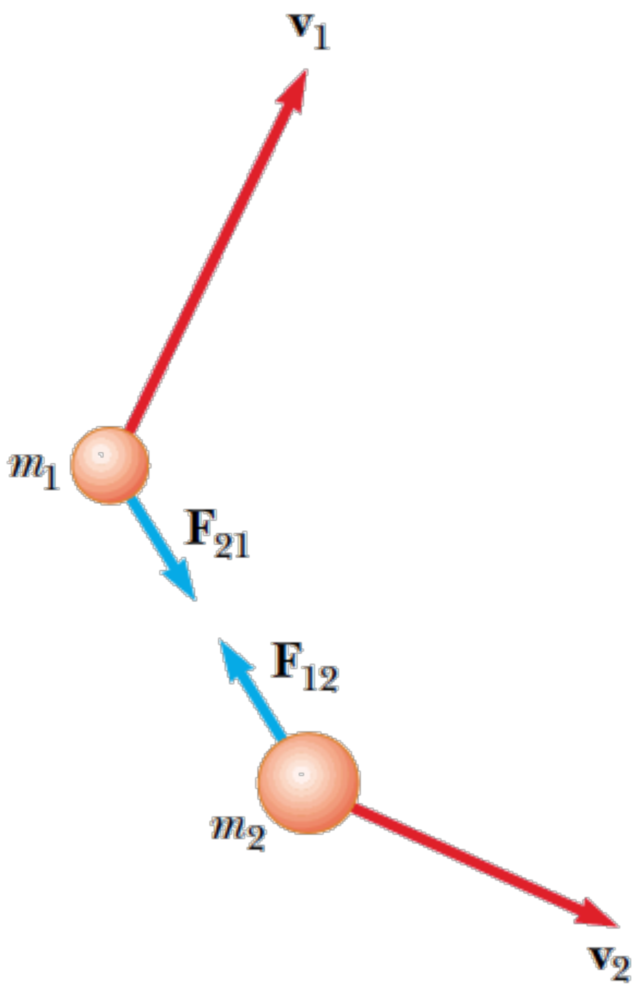


From Newton's third law:

$$\vec{F}_{12} = -\vec{F}_{21} \Rightarrow \vec{F}_{12} + \vec{F}_{21} = 0$$

We use the acceleration instead of forces:

$$m_1\vec{a}_1 + m_2\vec{a}_2 = 0$$

From the definition of acceleration:

$$m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} = 0$$

$$\frac{d(m_1\vec{v}_1)}{dt} + \frac{d(m_2\vec{v}_2)}{dt} = 0$$

$$\frac{d(m_1\vec{v}_1 + m_2\vec{v}_2)}{dt} = 0$$

9.3. Linear Momentum

For a closed system:

$$\frac{d(m_1\vec{v}_1 + m_2\vec{v}_2)}{dt} = 0 \Rightarrow m_1\vec{v}_1 + m_2\vec{v}_2 = \text{constant}$$

The linear momentum of a particle is a vector quantity that is defined as:

$$\boxed{\vec{p} = m\vec{v}}$$

in which m is the mass of the particle and v its velocity vector.

$$p_x = mv_x \quad p_y = mv_y \quad p_z = mv_z$$

The SI unit for momentum is the kilogram-meter per second (kgm/s).

9.3. Linear Momentum

From the definition of linear momentum we can express the Newton's second law as:

$$\boxed{\sum F = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} = \frac{d\vec{p}}{dt}}$$

The time rate of change of the momentum of a particle is equal to the net force acting on the particle and is in the direction of that force.

If there is a net force acting on a particle, it's momentum will change

If the net frce zero on a particle, it's momentum is constant

9.4. Linear Momentum for a System of Particles

Consider a system of n particles, each with its own mass, velocity, and linear momentum. The system as a whole has a total linear momentum **P,** which is defined to be the vector sum of the individual particles' linear momenta:

$$\vec{p} = \vec{p}_1 + \vec{p}_2 + \ldots + \vec{p}_n$$

$$\vec{p} = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots + m_n\vec{v}_n$$

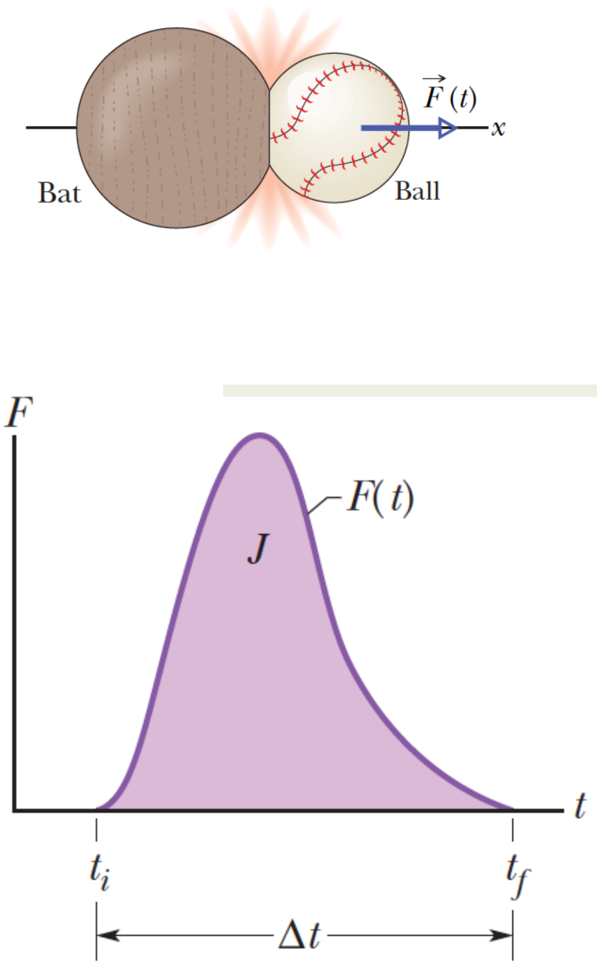
$$\vec{p} = M\vec{v}_{CM}$$

The linear momentum of a system of particles is equal to the product of the total
mass M of the system and the velocity of the center of mass.

The first derivation of this equation with time:

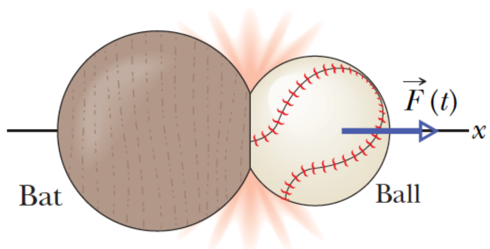
$$\frac{d\vec{p}}{dt} = M\frac{d\vec{v}_{CM}}{dt} = M\vec{a}_{CM} = \vec{F}_{net}$$

+ 9.5. Collision and Impuls



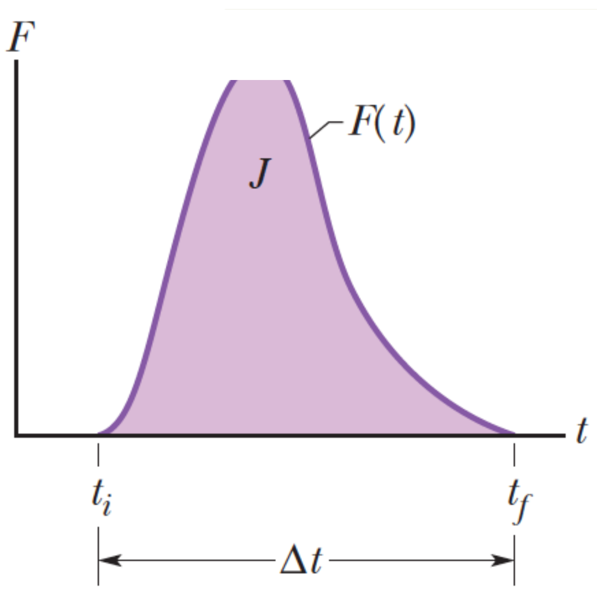
9.5. Collision and Impuls

Single Collision



Thus, in time interval dt, the change in the ball's momentum is:

$$d\vec{p} = \vec{F}(t)dt$$



The momentum change during the collision:

$$\int_{t_1}^{t_2} d\vec{p} = \int_{t_1}^{t_2} \vec{F}(t)dt$$

$$\vec{p}_s - \vec{p}_i = \Delta\vec{p} = \int_{t_1}^{t_2} \vec{F}(t)dt$$

9.5. Collision and Impuls

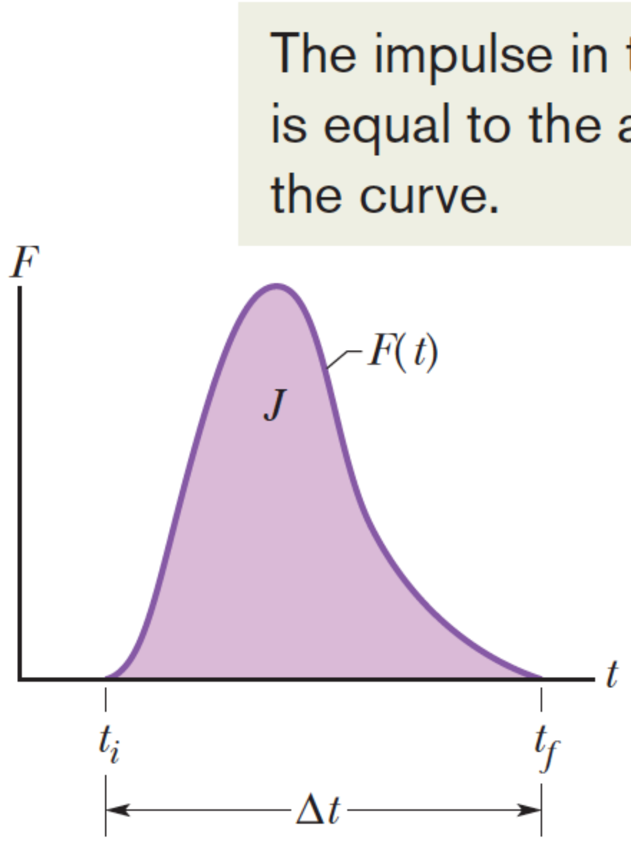
The momentum change for a given collision time interval is defined as impuls and shown as **J**

$$\boxed{\vec{J} = \vec{p}_s - \vec{p}_i = \Delta\vec{p} = \int_{t_1}^{t_2} \vec{F}(t)\,dt}$$

This equation is known as impuls-momentum theorem.

Impuls is a vector quantity.

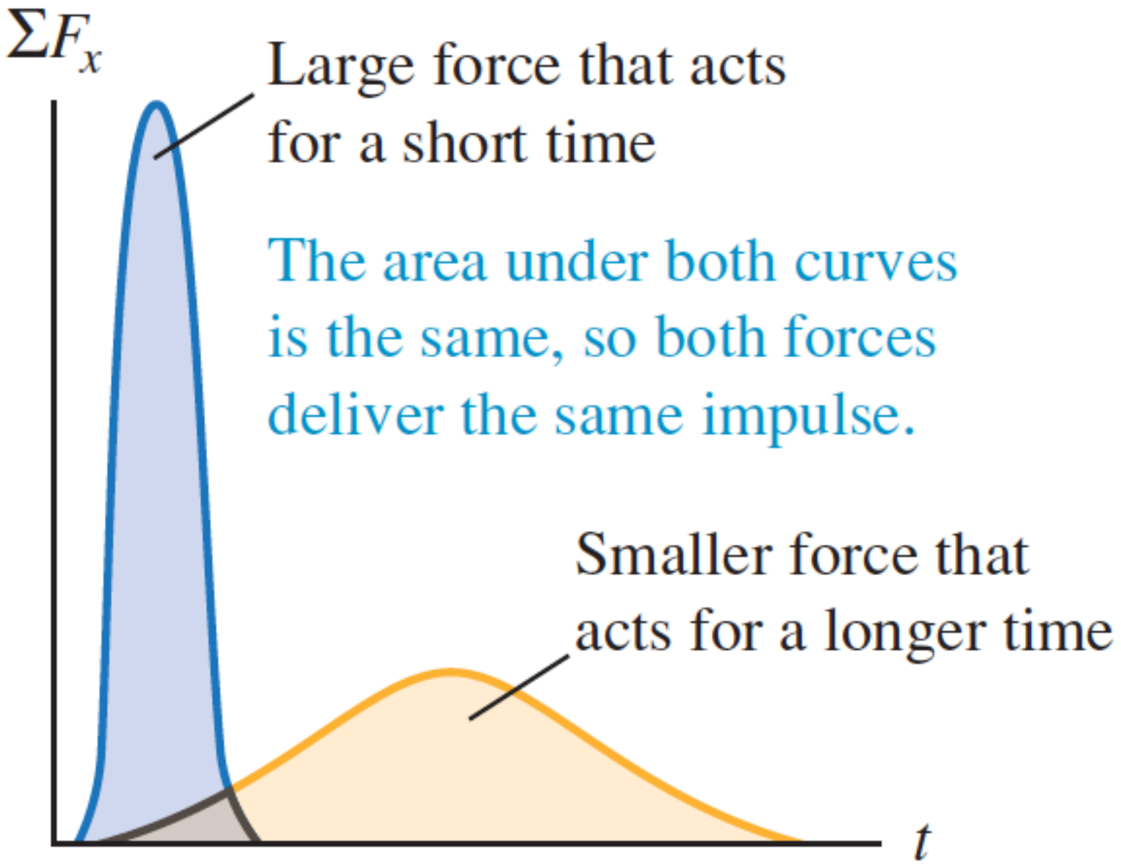
9.5. Collision and Impuls



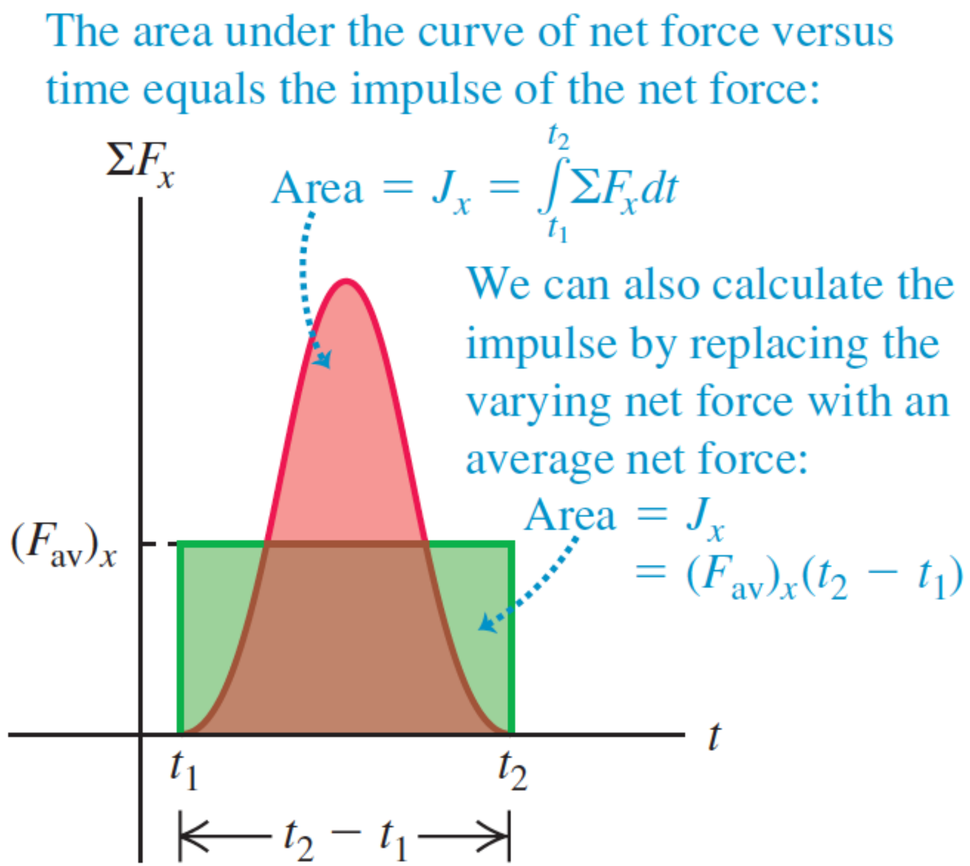
$$\boxed{\vec{J} = \int_{t_1}^{t_2} \vec{F}(t)\,dt}$$

İf thw force is constant, then impuls is given by: $\boxed{\vec{J} = \vec{F}\Delta t}$

9.5. Collision and Impuls



9.5. Collision and Impuls



$$\boxed{\vec{J} = \vec{F}_{avg.}\Delta t}$$

9.6. Conservation of Linear Momentum

For a isolated system, the net force, acting on particles of the systems, is zero:

$$\sum F_{net} = \frac{d\vec{p}}{dt} = 0$$

Therefor the momentum of an isolated system is

$$\vec{p} = \text{constant}$$

If no net external force acts on a system of particles, the total linear momentum of the system cannot change

This result is called **the law of conservation of linear momentum**. It can also be written as

$$\boxed{\vec{p}_i = \vec{p}_s}$$

9.6. Conservation of Linear Momentum

This result is called **the law of conservation of linear momentum**. It can also be written as

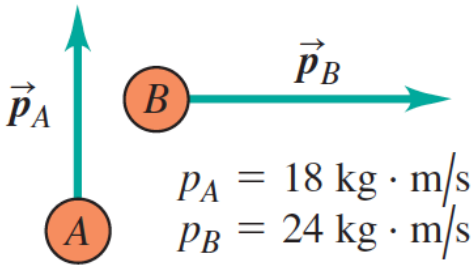
$$\boxed{\vec{p}_i = \vec{p}_f}$$

In words, this equation says that, for a closed, isolated system, total momentum at some initial time t_i is equal to total momentum at some later time t_f

The momentum here are the total momentum of sytems !

9.6. Conservation of Linear Momentum

Caution: Momentum is a vector quantity.



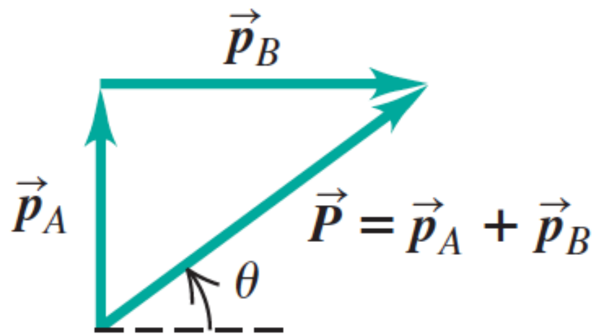
$p_A = 18\ \text{kg} \cdot \text{m/s}$
$p_B = 24\ \text{kg} \cdot \text{m/s}$

A system of two particles with momenta in different directions

You CANNOT find the magnitude of the total momentum by adding the magnitudes of the individual momenta!

$P = p_A + p_B = 42\ \text{kg} \cdot \text{m/s}$ ◀ WRONG

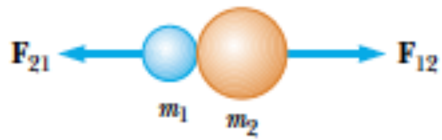
Instead, use vector addition:



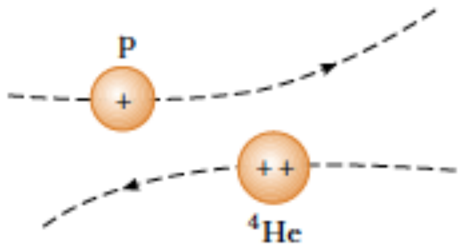
◀ RIGHT!

$$P = |\vec{p}_A + \vec{p}_B|$$
$$= 30\ \text{kg} \cdot \text{m/s at } \theta = 37°$$

9.7. Momentum and Kinetic Energy in Collision



(a)



- **Elastic Collision:**

The total kinetic energy (as well as total momentum) of the system is the same before and after the collision.

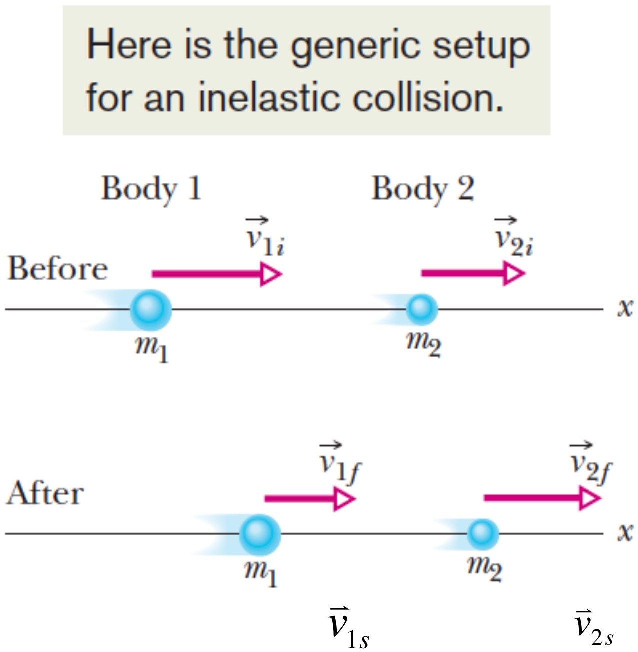
- **Inelastic Collision**

The total kinetic energy of the system is not the same before and after the collision (even though the momentum of the system is conserved)

- **Completely inelastic Collision**

When the colliding objects stick together after the collision, as happens when a meteorite collides with the Earth, the collision is called perfectly inelastic

9.8. Inelastic Collision in One Dimension



Since the momentum of system is conserved:

$$\vec{p}_{after} = \vec{p}_{initial}$$

In one dimension:

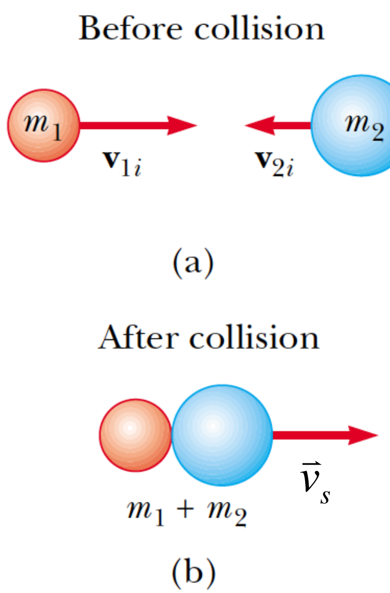
$$m_1 v_{1s} + m_2 v_{2s} = m_1 v_{1i} + m_2 v_{2i}$$

But the kinetic energy is **not conserved !**

$$KE_{after} \neq KE_{intial}$$

9.8. Inelastic Collision in One Dimension

Completely inelastic collision



From the conservation of momentum:

$$(m_1 + m_2)v_s = m_1 v_{1i} + m_2 v_{2i}$$

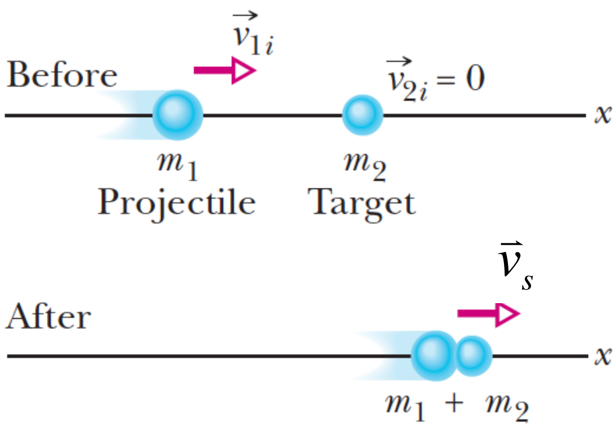
$$v_s = \frac{m_1 v_{1i} + m_2 v_{2i}}{(m_1 + m_2)}$$

KE is not conserved !

9.8. Inelastic Collision in One Dimension

Completely inelastic collision

In a completely inelastic collision, the bodies stick together.



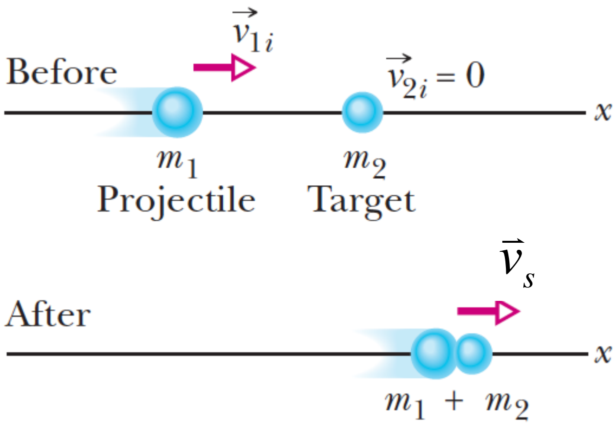
$$(m_1 + m_2)v_s = m_1 v_{1i}$$

$$v_s = \frac{m_1 v_{1i}}{(m_1 + m_2)}$$

9.8. Inelastic Collision in One Dimension

Completely inelastic collision

In a completely inelastic collision, the bodies stick together.



$$\vec{p} = M\vec{v}_{CM}$$

$$\vec{p} = \vec{p}_{1i} + \vec{p}_{2i}$$

$$\text{Constant ☺} \leftarrow \vec{v}_{CM} = \frac{\vec{p}}{M} = \frac{\vec{p}_{1i} + \vec{p}_{2i}}{M}$$

Constant

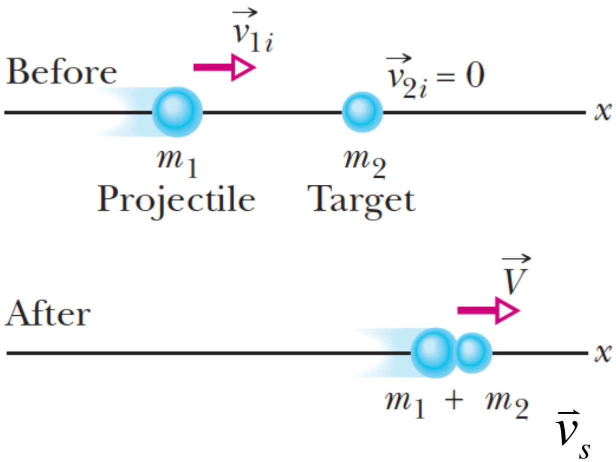
Constant

9.8. Inelastic Collision in One Dimension

Completely inelastic collision

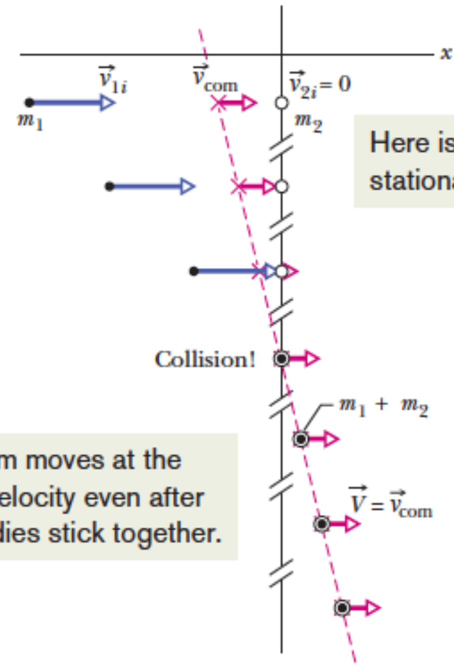
In a completely inelastic collision, the bodies stick together.

$$\vec{v}_{CM} = \frac{\vec{p}}{M} = \frac{\vec{p}_{1i}}{M}$$



The com of the two bodies is between them and moves at a constant velocity.

Here is the incoming projectile.

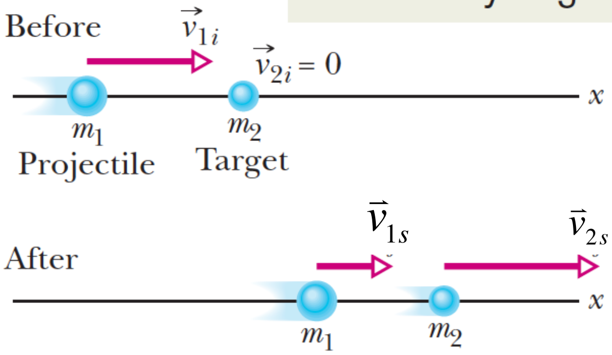


Here is the stationary target

The com moves at the same velocity even after the bodies stick together.

9.9. Elastic Collision in One Dimension

Here is the generic setup for an elastic collision with a stationary target.



From the conservation of momentum:

$$m_1 v_{1i} = m_1 v_{1s} + m_2 v_{2s}$$

From the conservation of KE

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1s}^2 + \frac{1}{2} m_2 v_{2s}^2$$

From these two equation we can obtain:

$$\boxed{v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}}$$

$$\boxed{v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}}$$

9.9. Elastic Collision in One Dimension

$$\boxed{v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}}$$

$$\boxed{v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}}$$

For $\boxed{m_1 >> m_2}$

$$v_{1f} \approx v_{1i} \qquad v_{2f} \approx 2v_{1i}$$

For $\boxed{m_2 >> m_1}$

$$v_{1s} \approx -v_{1i} \qquad v_{2f} \approx \frac{2m_1}{m_2} v_{1i}$$

For $\boxed{m_2 = m_1}$

$$v_{1f} = 0 \qquad v_{2f} = v_{1i}$$

9.9. Elastic Collision in One Dimension

Here is the generic setup for an elastic collision with a moving target.

From the conservation of momentum:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

From the conservation of KE

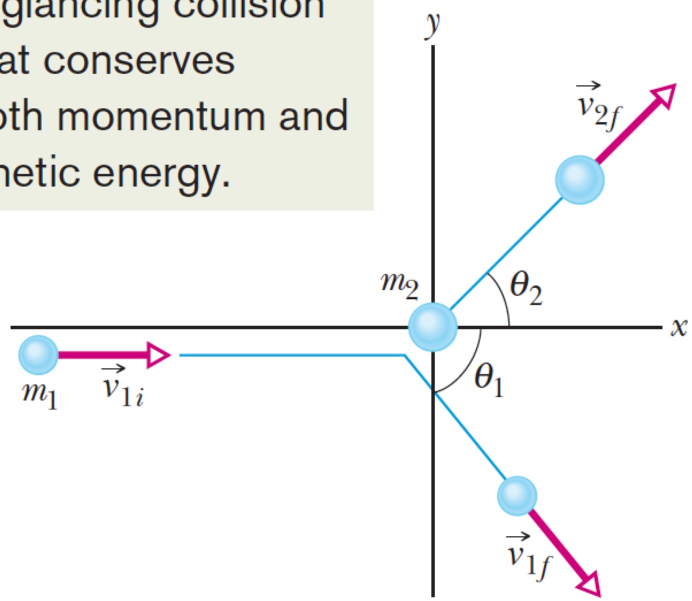
$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

$$\boxed{v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}}$$

$$\boxed{v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i}}$$

9.10. Collision in Two Dimensions

A glancing collision that conserves both momentum and kinetic energy.



The conservatio of momentum in vector form:

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

If the collison is elactic, then

$$K_{1i} + K_{2i} = K_{1f} + K_{2f}$$

The conservation of momentum in x-direction:

$$m_1 v_{1i} = m_1 v_{1s} \cos\theta_1 + m_2 v_{2s} \cos\theta_2$$

The conservation of momentum in y-direction:

$$0 = -m_1 v_{1s} \sin\theta_1 + m_2 v_{2s} \sin\theta_2$$