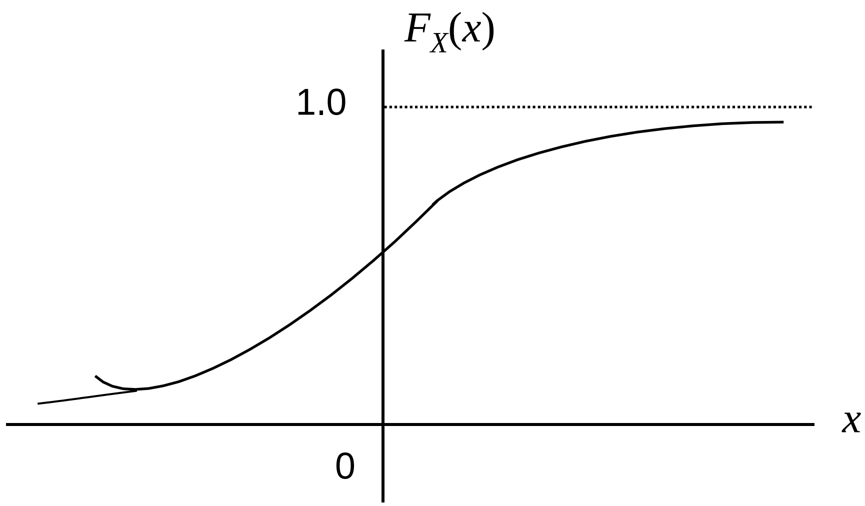
Kümülatif Dağılım Fonksiyonu (Sürekli)

- X sürekli bir rastgele değişken olsun. Bu durumda kümülatif dağılım fonksiyonu şu sekilde tanımlanır.

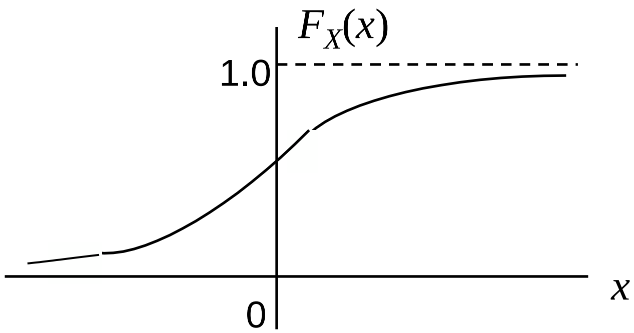
$$F_X(x) = \Pr[X \leq x]$$

- Tipik bir KDF şu şekilde görünür:



Kümülatif Dağılım Fonksiyonu (kdf), $F_X(x)$

$$F_X(x) = \Pr[X \leq x]$$



<u>Özellikler</u>

1. $0 \leq F_X(x) \leq 1$
2. $\lim_{x\to\infty} F_X(x) = 1; \quad \lim_{x\to-\infty} F_X(x) = 0$
3. Herhangi bir $\Delta x > 0$ için monotonik artan fonksiyon

$$F_X(x+\Delta x) \geq F_X(x)$$

4. $\Pr[a < X \leq b] = F_X(b) - F_X(a)$
5. $h > 0$ için $F_X(x)$ sağdan süreklidir

$$F_X(a) = \lim_{h\to 0} F_X(a+h) = F_X(a^+)$$

6. $\lim_{\Delta x\to 0} \Pr[a < X \leq a+\Delta x] = \lim_{\Delta x\to 0}[F_X(a+\Delta x) - F_X(a)] = 0$

Olasılık Yoğunluk Fonksiyonu (oyf), $f_X(x)$

$$f_X(x) = \frac{dF_X(x)}{dx}$$

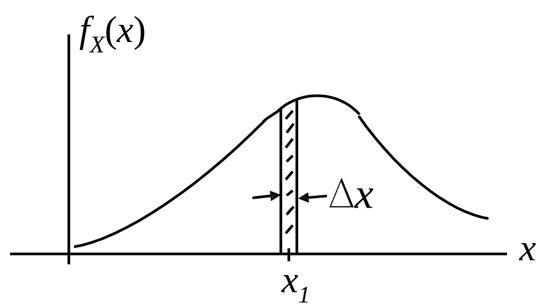
$$F_X(x) = \Pr[X \le x] = \int_{-\infty}^{x} f_X(z)dz$$

$$\Pr[a < X \le b] = \int_{a}^{b} f_X(x)dx$$

Özellikler:

$$f_X(x) \ge 0$$

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

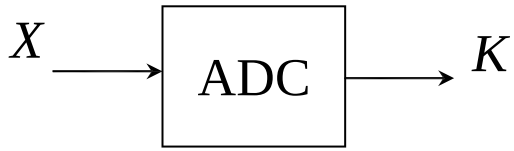


$$\Pr[a < X \le a + \Delta x] = \int_{a}^{a+\Delta x} f_X(x) \quad dx \approx f_X(a) \cdot \Delta x$$

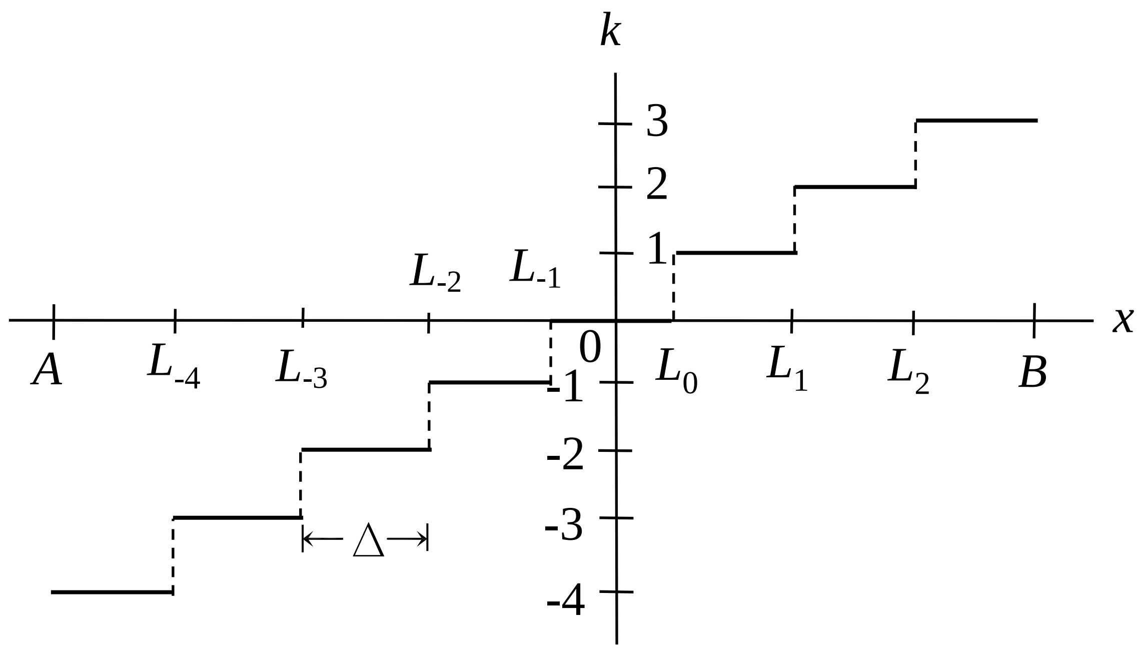
Dikkat $\lim_{\Delta x \to 0} \Pr[a < X \le a + \Delta x] = 0$

Sürekli ve Kesikli Rastgele Değişkenler

- Sürekli rastgele değişken X gibi bir girişe sahip olan bir analog sayısal çeviriciyi düşünün:



  - Bu durumda çıkış, K, girişe karşılık gelen ve sonlu değerlerde örneklenmiş kesikli rastgele değişken olur.



OYF ve OKF:

- OKF yaklaşık olarak şu şekilde yazılabilir:

$$\mathrm{f}_K[k] = \Pr[K = k] = \Pr[L_k < X \le L_k + \Delta] = \int_{L_k}^{L_k+\Delta} f_X(x)dx$$

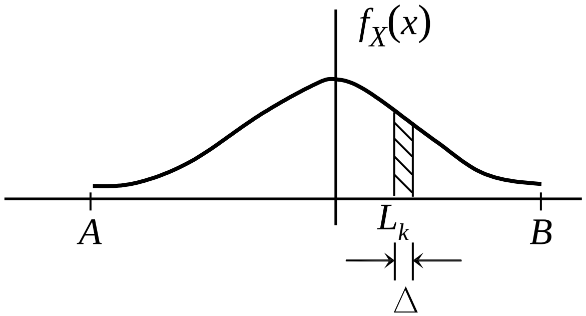
- Küçük pozitif Δ için

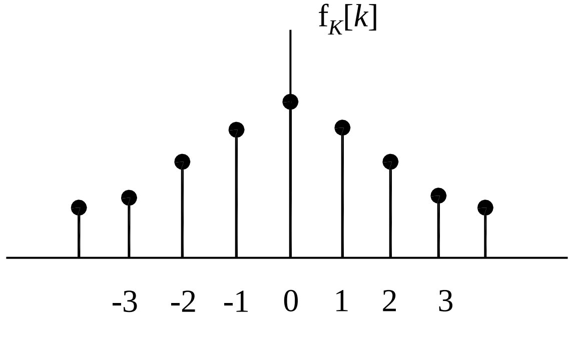
$$\mathrm{f}_K[k] \approx f_X(x)\,\Delta$$

– Sonuç olarak, $\mathrm{f}_K[k] \le 1$ için, $f_X(x)$ 1'den büyük değerler alabilir; ancak, her ikisi için de hala aşağıdaki eşitlikler sağlanmalıdır:

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

$$\sum_{k=-\infty}^{\infty} \mathrm{f}_K[k] = 1$$



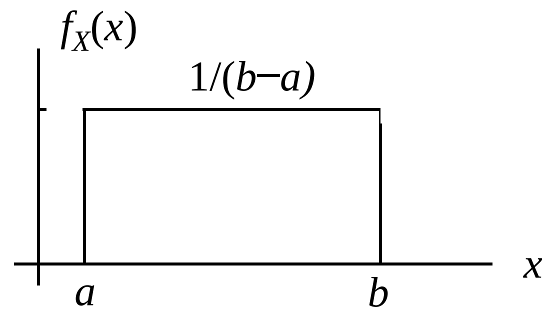


Bazı Sürekli Rastgele Değişkenler

Birbiçim Rastgele Değişken: parametreler *a*, *b*

Olasılık Yoğunluk Fonksiyonu:

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & diğer \end{cases}$$

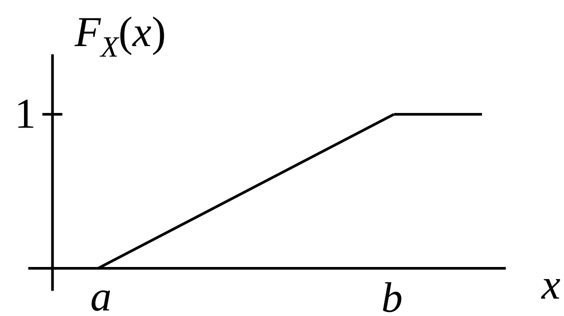


$$\int_a^b f_X(x)dx = 1$$

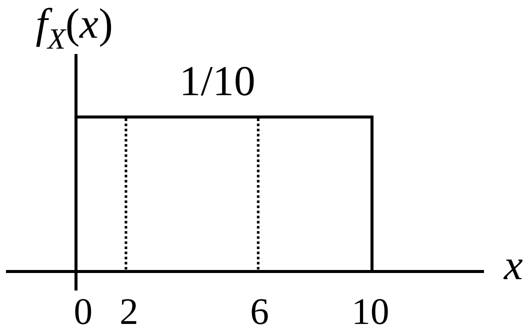
Kümülatif Dağılım Foksiyonu:

$$F_X(x) = \Pr[X \le x] = \int_{-\infty}^{x} f_x(z)dz \int_a^x \frac{1}{b-a} dz$$

$$F_X(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x < b \\ 1, & x \ge b \end{cases}$$



Örnek: $a = 0$, $b = 10$ ise $\Pr[2 < X \leq 6]$ nedir?



$$\Pr[2 < X \leq 6] = \int_2^6 f_X(x)dx = \int_2^6 \frac{1}{10}dx = \frac{4}{10}$$

Alternatif olarak,

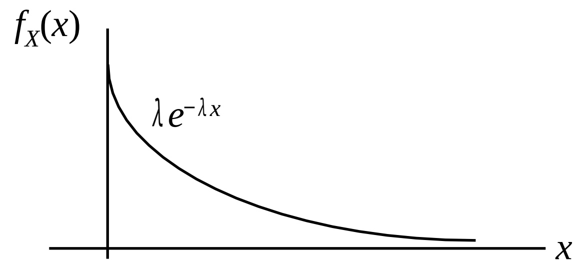
$$\Pr\left[2 < X \leq 6\right] = F_X(6) - F_X(2) = 0.6 - 0.2 = 0.4$$

Üstel Rastgele Değişken: parametre λ

Olasılık Yoğunluk Fonksiyonu:

$$f_X(x) = \begin{cases} 0, & x < 0 \\ \lambda e^{-\lambda x}, & x \geq 0 \end{cases}$$

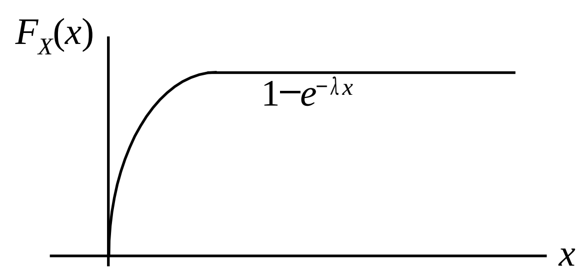
$$\int_0^{\infty} f_X(x)dx = \lambda \int_0^{\infty} e^{-\lambda x} dx = 1$$



Kümülatif Dağılım Fonksiyonu:

$$F_X(x) = \Pr[X \leq x] = \int_{-\infty}^{x} f_X(z)dz = \lambda \int_0^{x} e^{-\lambda z} dz = -e^{-\lambda z} \Big|_0^x = 1 - e^{-\lambda x}$$

$$F_X(x) \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$



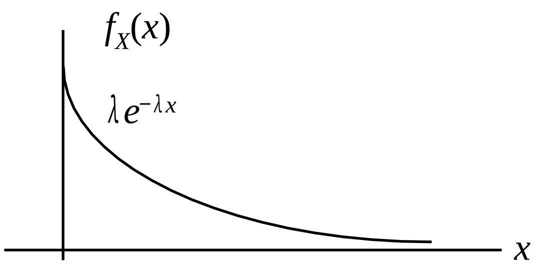
Dikkat: $\Pr[X > x] = 1 - \Pr[X \leq x] = e^{-\lambda x}$

<u>Örnek</u>:

Yazıcı kuyruğunda bekleme süresi üstel bir rastgele değişkendir.

$X \hat{=}$ bekleme zamanı

$f_X(x) = \lambda e^{-\lambda x} \qquad x \geq 0$



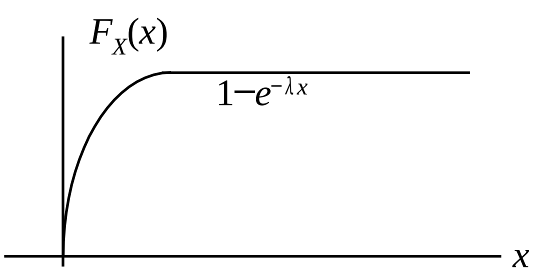
(a) $\Pr\left[T < X \leq 2T\right]$

$$= \int_T^{2T} \lambda e^{-\lambda u}\, du = e^{-\lambda T} - e^{-2\lambda T}$$

(b) $\Pr\left[T < X \leq 2T\right] = F_X(2T) - F_X(T)$

$$= 1 - e^{-2\lambda T} - 1 + e^{-\lambda T}$$

$$= e^{-\lambda T} - e^{-2\lambda T}$$



$\lambda = \frac{1}{T}$. ise

$\Pr\left[T < X \leq 2T\right] = e^{-1} - e^{-2} \cong 0.233$

Hafızasızlık Özelliği

t saniyesinde çalışır durumda olan bir sistem için bu zamandan sonra *h* saniye daha çalışır durumda olması olasılığı toplam *h* saniye çalışır durumda olması olasılığı ile aynıdır.

- Sıralama (queuing) teorisinin uygulamaları; habelerşme ve bilgisayar ağları

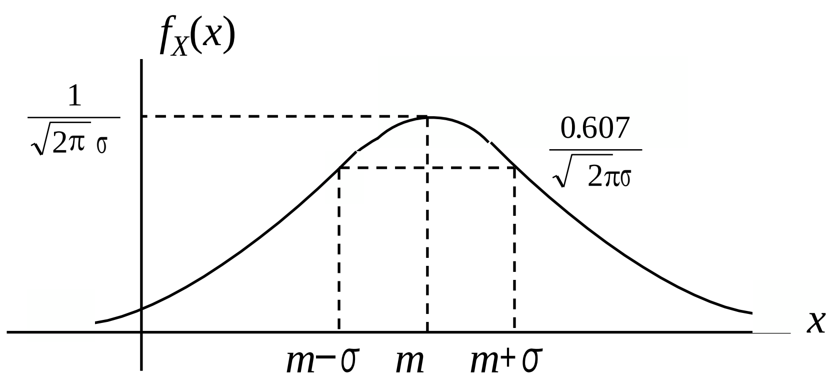
$$\Pr\left[X > t+h \mid X > t\right] = \frac{\Pr\left[\{X > t+h\}\cdot\{X > t\}\right]}{\Pr\left[\{X > t\}\right]} = \frac{\Pr\left[\{X > t+h\}\right]}{\Pr\left[\{X > t\}\right]}$$

$$= \frac{e^{-\lambda(t+h)}}{e^{-\lambda t}} = e^{-\lambda h} = \Pr\left[X > h\right]$$

Gausyen Rastgele Değişken: parametreler m, σ

Olasılık Yoğunluk Fonksiyonu:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-m)^2}{2\sigma^2}}, \qquad -\infty < x < \infty$$

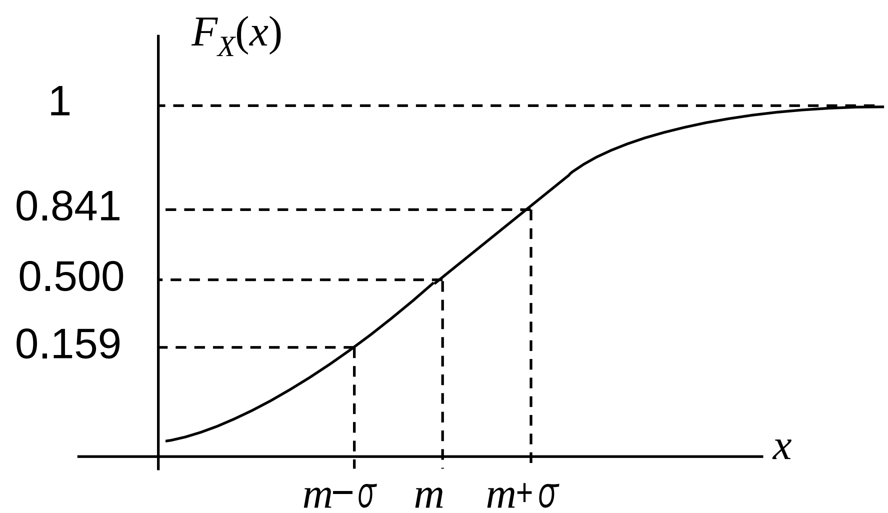


Kümülatif Dağılım Fonksiyonu:

$$F_X(x) = \Pr[X \le x] = \int_{-\infty}^{x} f_X(z)\,dz$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} e^{-\frac{(z-m)^2}{2\sigma^2}}\,dz$$

0.682



Gausyen Rastgele Değişkenler İçin Olasılıkların Bulunması

$$F_X(x) = \Pr[X \le x] = \Phi\left(\frac{x-m}{\sigma}\right) = 1 - Q\left(\frac{x-m}{\sigma}\right)$$

$$\Phi(y) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{z^2}{2}dz} \qquad Q(y) = 1 - \Phi(y) = \Phi(y) \frac{1}{\sqrt{2\pi}} \int_{y}^{\infty} e^{-\frac{z^2}{2}dz}$$

Gausyen yoğunluk simetrik olduğundan:

$$\Phi(-y) = Q(y) \qquad Q(-z) = 1 - Q(z).$$

Bu fonksiyonlar tablolar halinde sunulmuştur.

Yaklaştırma: $y \geq 2$ olduğunda Q fonksiyonu söyle yazılabilir

$$Q(y) = \frac{1}{\sqrt{2\pi}} \int_{y}^{\infty} e^{-z^2/2} dz \approx \frac{1}{\sqrt{2\pi}\, y} e^{-y^2/2}$$

Tablolar

Table of Q function values

x	$Q(x)$	x	$Q(x)$	x	$Q(x)$	x	$Q(x)$
0.00	5.0000E-01	1.00	1.5866E-01	2.00	2.2750E-02	5.00	2.8665E-07
0.02	4.9202E-01	1.02	1.5386E-01	2.05	2.0182E-02	5.10	1.6983E-07
0.04	4.8405E-01	1.04	1.4917E-01	2.10	1.7864E-02	5.20	9.9644E-08
0.06	4.7608E-01	1.06	1.4457E-01	2.15	1.5778E-02	5.30	5.7901E-08
0.08	4.6812E-01	1.08	1.4007E-01	2.20	1.3903E-02	5.40	3.3320E-08
0.10	4.6017E-01	1.10	1.3567E-01	2.25	1.2224E-02	5.50	1.8990E-08
0.12	4.5224E-01	1.12	1.3136E-01	2.30	1.0724E-02	5.60	1.0718E-08
0.14	4.4433E-01	1.14	1.2714E-01	2.35	9.3867E-03	5.70	5.9904E-09
0.16	4.3644E-01	1.16	1.2302E-01	2.40	8.1975E-03	5.80	3.3157E-09
0.18	4.2858E-01	1.18	1.1900E-01	2.45	7.1428E-03	5.90	1.8175E-09
0.20	4.2074E-01	1.20	1.1507E-01	2.50	6.2097E-03	6.00	9.8659E-10
0.22	4.1294E-01	1.22	1.1123E-01	2.55	5.3861E-03	6.10	5.3034E-10
0.24	4.0517E-01	1.24	1.0749E-01	2.60	4.6612E-03	6.20	2.8232E-10
0.26	3.9743E-01	1.26	1.0383E-01	2.65	4.0246E-03	6.30	1.4882E-10
0.28	3.8974E-01	1.28	1.0027E-01	2.70	3.4670E-03	6.40	7.7688E-11
0.30	3.8209E-01	1.30	9.6800E-02	2.75	2.9798E-03	6.50	4.0160E-11
0.32	3.7448E-01	1.32	9.3418E-02	2.80	2.5551E-03	6.60	2.0558E-11
0.34	3.6693E-01	1.34	9.0123E-02	2.85	2.1860E-03	6.70	1.0421E-11
0.36	3.5942E-01	1.36	8.6915E-02	2.90	1.8658E-03	6.80	5.2310E-12
0.38	3.5197E-01	1.38	8.3793E-02	2.95	1.5889E-03	6.90	2.6001E-12
0.40	3.4458E-01	1.40	8.0757E-02	3.00	1.3499E-03	7.00	1.2798E-12
0.42	3.3724E-01	1.42	7.7804E-02	3.05	1.1442E-03	7.10	6.2378E-13
0.44	3.2997E-01	1.44	7.4934E-02	3.10	9.6760E-04	7.20	3.0106E-13
0.46	3.2276E-01	1.46	7.2145E-02	3.15	8.1635E-04	7.30	1.4388E-13
0.48	3.1561E-01	1.48	6.9437E-02	3.20	6.8714E-04	7.40	6.8092E-14
0.50	3.0854E-01	1.50	6.6807E-02	3.25	5.7703E-04	7.50	3.1909E-14
0.52	3.0153E-01	1.52	6.4255E-02	3.30	4.8342E-04	7.60	1.4807E-14
0.54	2.9460E-01	1.54	6.1780E-02	3.35	4.0406E-04	7.70	6.8033E-15
0.56	2.8774E-01	1.56	5.9380E-02	3.40	3.3693E-04	7.80	3.0954E-15
0.58	2.8096E-01	1.58	5.7053E-02	3.45	2.8029E-04	7.90	1.3945E-15
0.60	2.7425E-01	1.60	5.4799E-02	3.50	2.3263E-04	8.00	6.2210E-16
0.62	2.6763E-01	1.62	5.2616E-02	3.55	1.9262E-04	8.10	2.7480E-16
0.64	2.6109E-01	1.64	5.0503E-02	3.60	1.5911E-04	8.20	1.2019E-16
0.66	2.5463E-01	1.66	4.8457E-02	3.65	1.3112E-04	8.30	5.2056E-17
0.68	2.4825E-01	1.68	4.6479E-02	3.70	1.0780E-04	8.40	2.2324E-17
0.70	2.4196E-01	1.70	4.4565E-02	3.75	8.8417E-05	8.50	9.4795E-18
0.72	2.3576E-01	1.72	4.2716E-02	3.80	7.2348E-05	8.60	3.9858E-18
0.74	2.2965E-01	1.74	4.0930E-02	3.85	5.9059E-05	8.70	1.6594E-18
0.76	2.2363E-01	1.76	3.9204E-02	3.90	4.8096E-05	8.80	6.8408E-19
0.78	2.1770E-01	1.78	3.7538E-02	3.95	3.9076E-05	8.90	2.7923E-19
0.80	2.1186E-01	1.80	3.5930E-02	4.00	3.1671E-05	9.00	1.1286E-19
0.82	2.0611E-01	1.82	3.4380E-02	4.10	2.0658E-05	9.10	4.5166E-20
0.84	2.0045E-01	1.84	3.2884E-02	4.20	1.3346E-05	9.20	1.7897E-20
0.86	1.9489E-01	1.86	3.1443E-02	4.30	8.5399E-06	9.30	7.0223E-21
0.88	1.8943E-01	1.88	3.0054E-02	4.40	5.4125E-06	9.40	2.7282E-21
0.90	1.8406E-01	1.90	2.8717E-02	4.50	3.3977E-06	9.50	1.0495E-21
0.92	1.7879E-01	1.92	2.7429E-02	4.60	2.1125E-06	9.60	3.9972E-22
0.94	1.7361E-01	1.94	2.6190E-02	4.70	1.3008E-06	9.70	1.5075E-22
0.96	1.6853E-01	1.96	2.4998E-02	4.80	7.9333E-07	9.80	5.6293E-23
0.98	1.6354E-01	1.98	2.3852E-02	4.90	4.7918E-07	9.90	2.0814E-23

Table of inverse Q function values

l	$x=Q^{-1}(10^{-l})$	l	$x=Q^{-1}(10^{-l})$	l	$x=Q^{-1}(10^{-l})$	l	$x=Q^{-1}(10^{-l})$
1	1.2816	7	5.1993	13	7.3488	19	9.0133
2	2.3263	8	5.6120	14	7.6506	20	9.2623
3	3.0902	9	5.9978	15	7.9413	21	9.5050
4	3.7190	10	6.3613	16	8.2221	22	9.7418
5	4.2649	11	6.7060	17	8.4938	23	9.9730
6	4.7534	12	7.0345	18	8.7573	24	10.1990

Table of erf function values

x	erf(x)	x	erf(x)	x	erf(x)	x	erf(x)
0.00	0.00000	0.40	0.42839	0.80	0.74210	1.20	0.91031
0.01	0.01128	0.41	0.43797	0.81	0.74800	1.21	0.91296
0.02	0.02257	0.42	0.44747	0.82	0.75381	1.22	0.91553
0.03	0.03384	0.43	0.45689	0.83	0.75952	1.23	0.91805
0.04	0.04511	0.44	0.46623	0.84	0.76514	1.24	0.92051
0.05	0.05637	0.45	0.47548	0.85	0.77067	1.25	0.92290
0.06	0.06762	0.46	0.48466	0.86	0.77610	1.30	0.93401
0.07	0.07886	0.47	0.49375	0.87	0.78144	1.35	0.94376
0.08	0.09008	0.48	0.50275	0.88	0.78669	1.40	0.95229
0.09	0.10128	0.49	0.51167	0.89	0.79184	1.45	0.95970
0.10	0.11246	0.50	0.52050	0.90	0.79691	1.50	0.96611
0.11	0.12362	0.51	0.52924	0.91	0.80188	1.55	0.97162
0.12	0.13476	0.52	0.53790	0.92	0.80677	1.60	0.97635
0.13	0.14587	0.53	0.54646	0.93	0.81156	1.65	0.98038
0.14	0.15695	0.54	0.55494	0.94	0.81627	1.70	0.98379
0.15	0.16800	0.55	0.56332	0.95	0.82089	1.75	0.98667
0.16	0.17901	0.56	0.57162	0.96	0.82542	1.80	0.98909
0.17	0.18999	0.57	0.57982	0.97	0.82987	1.85	0.99111
0.18	0.20094	0.58	0.58792	0.98	0.83423	1.90	0.99279
0.19	0.21184	0.59	0.59594	0.99	0.83851	1.95	0.99418
0.20	0.22270	0.60	0.60386	1.00	0.84270	2.00	0.99532
0.21	0.23352	0.61	0.61168	1.01	0.84681	2.05	0.99626
0.22	0.24430	0.62	0.61941	1.02	0.85084	2.10	0.99702
0.23	0.25502	0.63	0.62705	1.03	0.85478	2.15	0.99764
0.24	0.26570	0.64	0.63459	1.04	0.85865	2.20	0.99814
0.25	0.27633	0.65	0.64203	1.05	0.86244	2.25	0.99854
0.26	0.28690	0.66	0.64938	1.06	0.86614	2.30	0.99886
0.27	0.29742	0.67	0.65663	1.07	0.86977	2.35	0.99911
0.28	0.30788	0.68	0.66378	1.08	0.87333	2.40	0.99931
0.29	0.31828	0.69	0.67084	1.09	0.87680	2.45	0.99947
0.30	0.32863	0.70	0.67780	1.10	0.88021	2.50	0.99959
0.31	0.33891	0.71	0.68467	1.11	0.88353	2.55	0.99969
0.32	0.34913	0.72	0.69143	1.12	0.88679	2.60	0.99976
0.33	0.35928	0.73	0.69810	1.13	0.88997	2.65	0.99982
0.34	0.36936	0.74	0.70468	1.14	0.89308	2.70	0.99987
0.35	0.37938	0.75	0.71116	1.15	0.89612	2.75	0.99990
0.36	0.38933	0.76	0.71754	1.16	0.89910	2.80	0.99992
0.37	0.39921	0.77	0.72382	1.17	0.90200	2.85	0.99994
0.38	0.40901	0.78	0.73001	1.18	0.90484	2.90	0.99996
0.39	0.41874	0.79	0.73610	1.19	0.90761	2.95	0.99997

<u>Örnek</u>:

X, ortalama değeri m, standart sapması σ olan Gausyen bir rastgele değişkendir.

a) $\Pr\left[m-2\sigma < X \le m+2\sigma\right]=?$

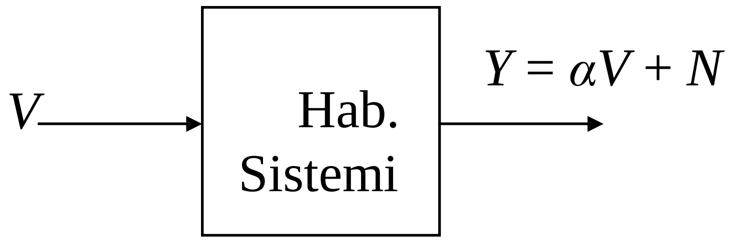
$$\Pr[m-2\sigma < X \le m+2\sigma] = 1-\Pr[X \le m-2\sigma]-\Pr[X > m+2\sigma]$$
$$= 1-2\Pr[X > m+2\sigma] = 1-2Q\left(\frac{m+2\sigma-m}{\sigma}\right)$$
$$= 1-2Q(2) = 1-4.56\times10^{-2} = 0.9544$$

b) $m = 5$ and variance $\sigma^2 = 64$ ise $\Pr[m - \sigma < X \le m + \sigma]=?$

$m = 5$ and $\sigma = 8$ için

$$\Pr\left[-3 < X \le 13\right] = 1-\Pr\left[X \le -3\right]-\Pr\left[X > 13\right] = 1-2\Pr\left[X > 13\right]$$
$$= \; = 1-2Q\left(\frac{13-5}{8}\right) = 1-2Q(1) = 1-2\times0.159 = 0.682$$

<u>Örnek</u>:



$\alpha = 10^{-2}$

N Gaussian r.d.

$m_N = 0$

$\sigma_N = 2$

(a) $V = 500$ ise $\Pr[Y \leq 0]$=?

$$\Pr[Y \leq 0] = \Pr[\alpha V + N \leq 0] = \Pr[N \leq -5]$$

$$= \Phi\left(\frac{-5}{\sigma_N}\right) = \Phi(2.5) = Q(2.5)$$

Q-fonksiyon tablosundan:

$$Q(2.5) = 6.21 \times 10^{-3}.$$

(b) $\Pr[Y \leq 0] = 10^{-6}$ yapacak V nedir?

$$\Pr\left[Y \leq 0\right] = \Pr\left[\alpha V + N \leq 0\right] = \Pr\left[N \leq -\alpha V\right] = 10^{-6}$$

$$\Phi\left(\frac{-\alpha V}{\sigma_N}\right) = Q\left(\frac{\alpha V}{\sigma_N}\right) = 10^{-6}$$

Ters Q-fonksiyonu tablosundan:

$$Q\left(\frac{\alpha V}{\sigma_N}\right) = 10^{-6} \Rightarrow \frac{\alpha V}{\sigma_N} = \frac{10^{-2} V}{2} = 4.7535$$

$$\therefore \quad V = 950.70$$

Gausyen r.d. ler İçin "Hata Fonksiyonu" nu Kullanarak Olasılıkların Belirlenmesi

İlişkiler:

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-m)^2}{2\sigma^2}} dz = \frac{1}{2}\left(1 + \text{erf}\left(\frac{x-m}{\sqrt{2}\sigma}\right)\right)$$

burada

$$\text{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-z^2} dz \qquad \text{ve} \qquad \text{erf}(-y) = -\text{erf}(y)$$

Zaman zaman kullanılan başka bir hata fonksiyonu tanımı:

$$\text{erfc}(y) = \frac{2}{\sqrt{\pi}} \int_y^{\infty} e^{-z^2} dz = \frac{1}{2} - \text{erf}(y)$$

Bu tanım Q fonksiyonu ile ilgilidir:

$$Q(y) = \frac{1}{2}\text{erfc}\left(\frac{y}{\sqrt{2}}\right)$$

Hem "erf" hem de "erfc" MATLAB da komut olarak mevcuttur

Örnek:

X, ortalama değeri m=1, varyansı σ^2=6.25 olan Gausyen bir rastgele değişkendir.

(a) $\Pr[X \leq 1.7]$ (b) $\Pr[X \leq 0.2]$ (c) $\Pr[X > 3.5]$

$$\text{(a)} \quad \Pr[X \leq 1.7] = \int_{-\infty}^{1.7} f_X(x)dx = \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{1.7-1}{2.5\sqrt{2}}\right)\right] = 0.6103$$

$$\text{(b)} \quad \Pr[X \leq 0.2] = \int_{-\infty}^{0.2} f_X(x)dx = \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{0.2-1}{2.5\sqrt{2}}\right)\right]$$

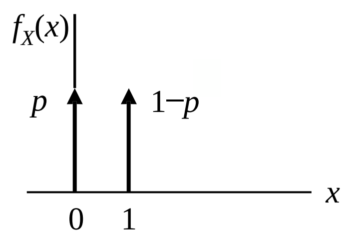
$$= \frac{1}{2}\left[1-\operatorname{erf}\left(\frac{1-0.2}{2.5\sqrt{2}}\right)\right] = 0.3745$$

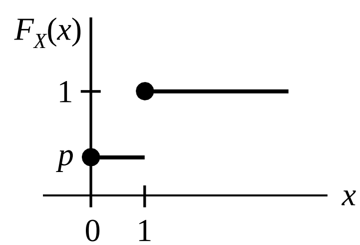
$$\text{(c)} \quad \Pr[X > 3.5] = 1 - \Pr[X \leq 3.5] = 1 - \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{3.5-1}{2.5\sqrt{2}}\right)\right]$$

$$= \frac{1}{2}\left[1-\operatorname{erf}\left(\frac{3.5-1}{2.5\sqrt{2}}\right)\right] = 0.1587$$

Kesikli Rastgele Değişken Gösterimi

$\delta\,(x)$ impalsları sıfır kalınlık sonlu olasılıkl sağlar





Karışık Rastgele Değişken

oyf de impalslar vardır ve oyf bir ya da birden fazla zaman aralığında süreklidir.