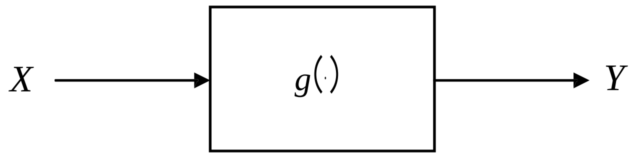
Rastgele Değişkenin Dönüşümleri

$Y = g(X)$ verildiğinde X rastgele ise Y de rastgeledir.

- Amaç $f_X(x)$ verildiğinde $f_Y(y)$ yi bulmaktır.



$$Y = g(X) \quad \text{and} \quad X = g^{-1}(Y)$$

- Sistem hafızasızdır: Y, sadece X in şu andaki değerlerine bağlıdır.

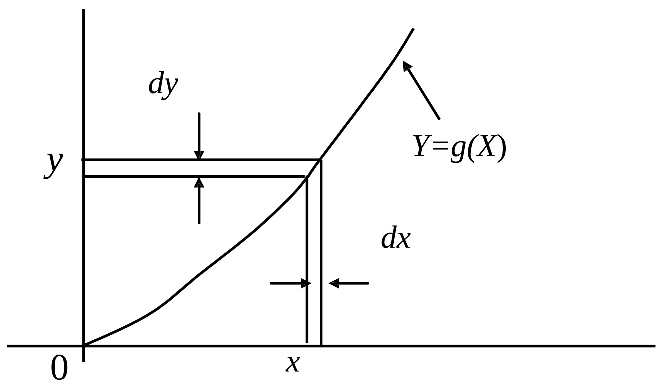
(örnek: $Y = 2X$)

– Giriş-çıkış ilişkisi lineer ya da nonlineer olabilir.

– $X = g^{-1}(Y)$ nin tersi çok değerli (multi-valued) olabilir.

Tek Değerli Durum

Transfer Karakteristiği



$$\Pr\left[y < Y \le y + dy\right] = \Pr\left[x < X \le x + dx\right]$$

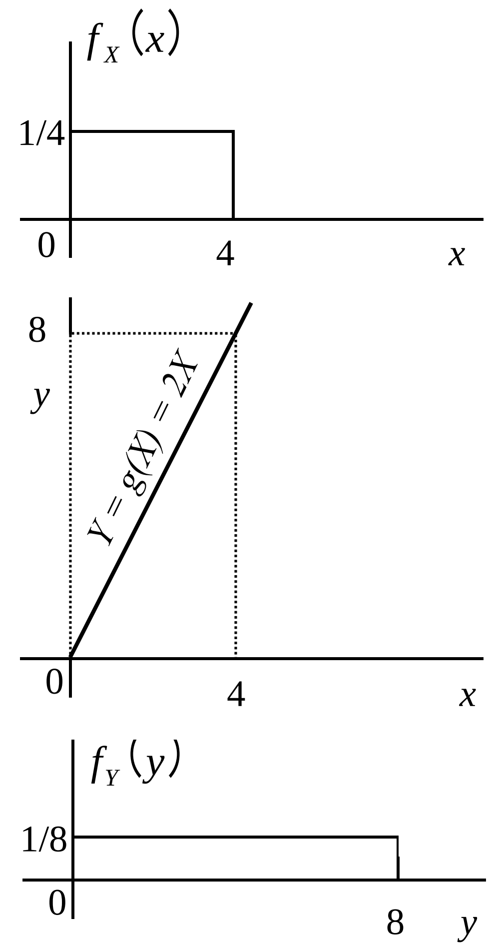
$$f_Y(y)\left|dy\right| = f_X(x)\left|dx\right|$$

$$f_Y(y) = \frac{1}{\left|dy/dx\right|} f_X(x)\Bigg|_{x=g^{-1}(y)}$$

<u>Örnek</u>:

$Y = g(X) = 2X$, ve X in oyf si aşağıdaki gibi olsun

$$f_X(x) = \begin{cases} \dfrac{1}{4}, & 0 \le X \le 4 \\ 0, & \text{diğer} \end{cases}$$



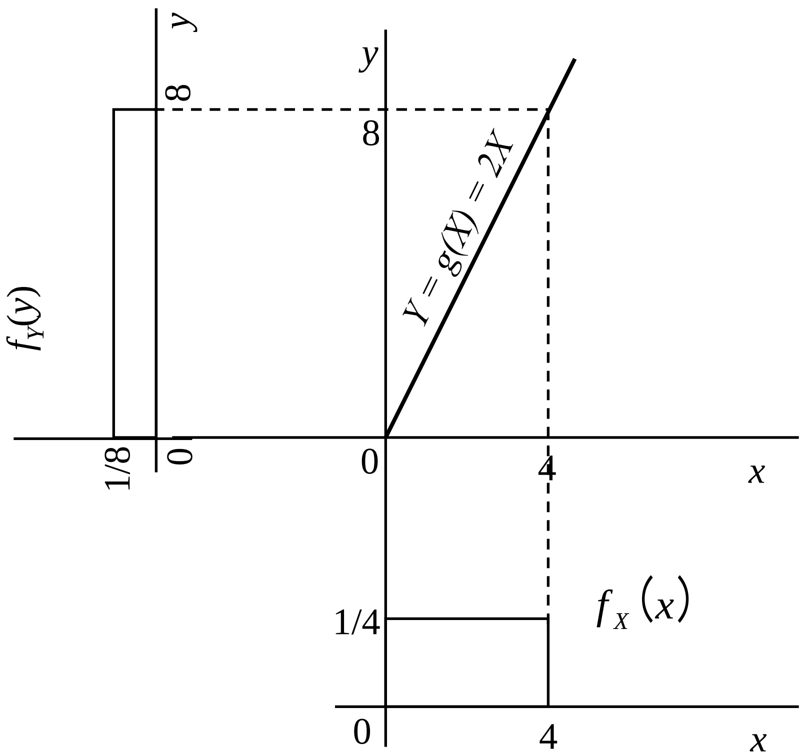
burada $\left|\dfrac{dy}{dx}\right| = 2$, $X = g^{-1}(Y) = \dfrac{1}{2}Y$ olur.

Y nin oyf si şu şekilde elde edilir:

$$f_Y(y) = \frac{1}{|dy/dx|} f_X(x)\Big|_{x=g^{-1}(y)}$$

$$= \begin{cases} \dfrac{1}{8}, & 0 \le Y \le 8 \\ 0, & \text{diğer} \end{cases}$$

Grafik Gösterim:



$$Y = aX$$

$$f_X(x) = \frac{1}{|a|} f_x\left(\frac{y}{a}\right)$$

<u>Örnek</u>:

$Y = g(X) = X+3$, ve X in oyf si aşağıdaki gibi olsun

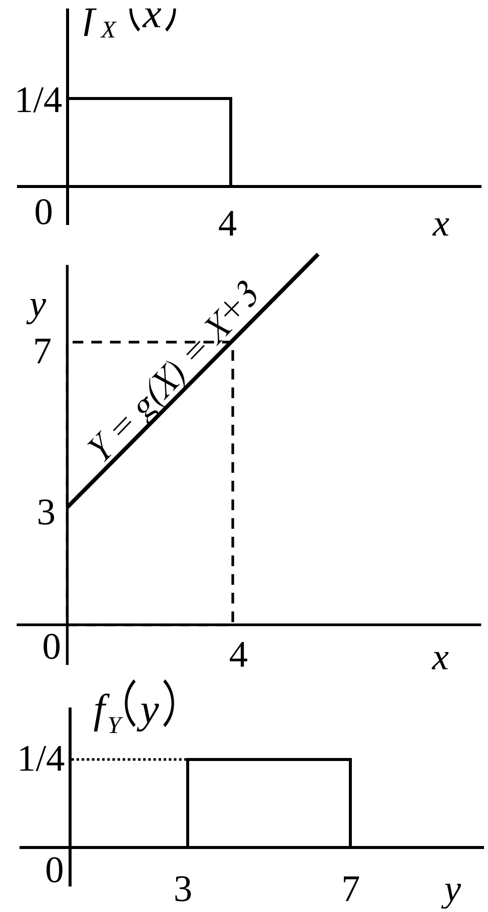
$$f_X(x) = \begin{cases} \dfrac{1}{4}, & 0 \le X \le 4 \\ 0, & \text{diğer} \end{cases}$$

Burada $\left|\dfrac{dy}{dx}\right| = 1,\ X = g^{-1}(Y) = Y - 3$ olur.

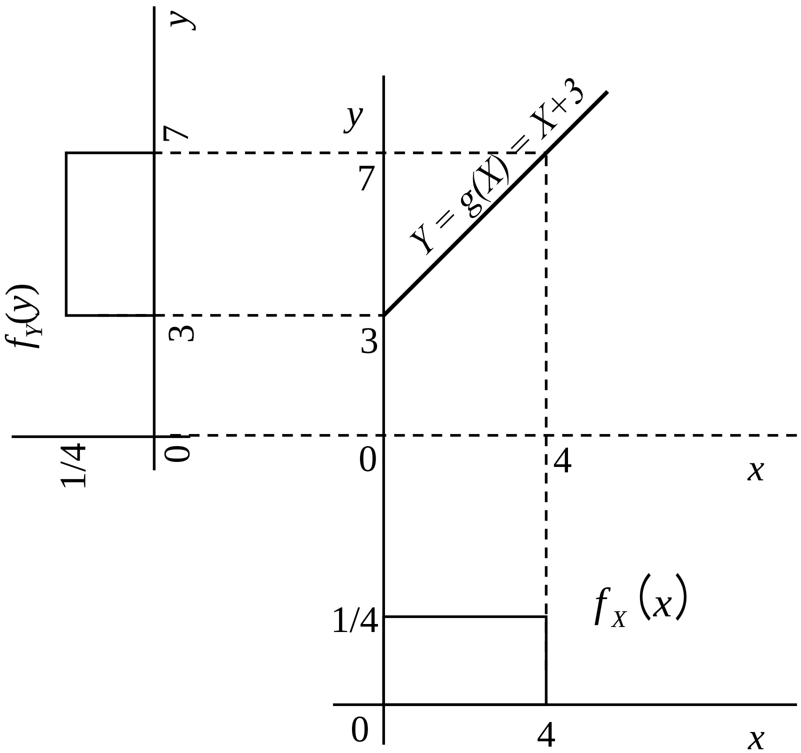
Y nin oyf si şu şekilde elde edilir:

$$f_Y(y) = \frac{1}{|dy/dx|} f_X(x)\Big|_{x=g^{-1}(y)}$$

$$= \begin{cases} \dfrac{1}{4}, & 3 \le Y \le 7 \\ 0, & \text{diğer} \end{cases}$$



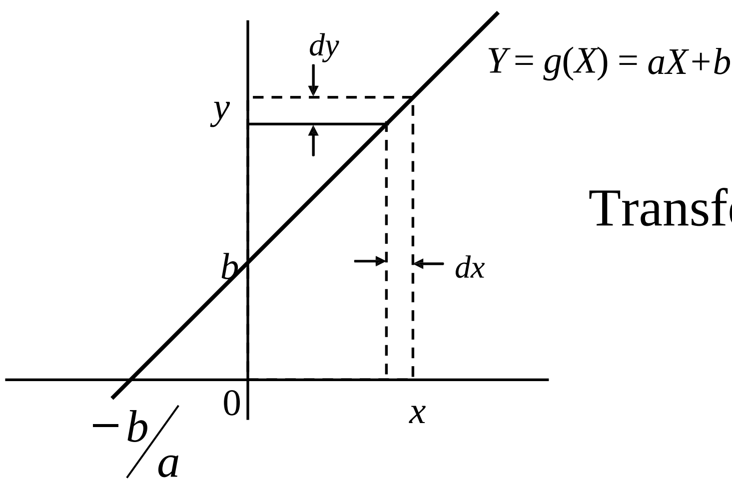
Grafik Gösterim:



Genelleme:

$Y = g(X) = aX + b$, ve girişin oyf si $f_X(x)$ verildiğinde $f_Y(y)$ bulma işlemi.

Burada $\left|\frac{dy}{dx}\right| = |a|, X = g^{-1}(Y) = \frac{Y-b}{a}$ olur.



Transfer Karakteristiği

$$f_Y(y) = \frac{1}{|dy/dx|} f_X(x)\Big|_{x=\frac{y-b}{a}} = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

Önemli Dönüşüm

$$Y = aX + b$$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

Özel Durumlar

$$Y = aX$$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y}{a}\right)$$

$$Y = X + b$$

$$f_Y(y) = f_X(y-b)$$

<u>Örnek</u>:

$f_X(x)$ Gausyen bir oyf olsun:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-m_X)^2}{2\sigma_X^2}} \qquad -\infty < x < \infty$$

buradan

$$f_Y(y) = \frac{1}{|a|} \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{((y-b)/a-m_X)^2}{2\sigma_X^2}} = \frac{1}{\sqrt{2\pi}\,|a|\,\sigma_X} e^{-\frac{(y-(b+am_X))^2}{2a^2\sigma_X^2}}$$

Yani, $f_Y(y)$ $m_Y = b + am_X$, $\sigma_Y = |a|\sigma_X$ olan bir Gausyen yoğunluktur.

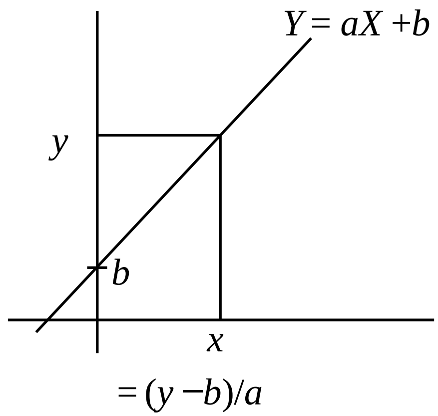
Alternatif Metod: Kümülatif Dağılım Fonksiyonu Kullanımı

Örnek:

$$Y = aX + b \qquad a > 0$$

$$F_Y(y) = \Pr[Y \le y] = \Pr[aX + b \le y]$$

$$= \Pr\left[X \le \frac{y-b}{a}\right] = F_X\left[\frac{y-b}{a}\right]$$



Buradan

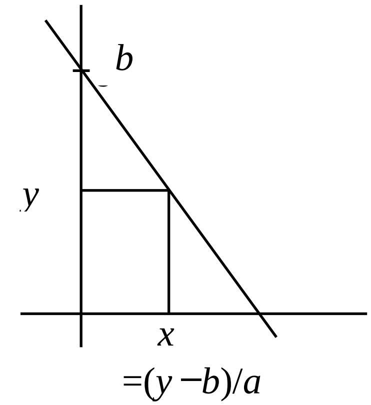
$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X\left(\frac{y-b}{a}\right) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

Örnek:

Yukarıdaki problemi $a < 0$ için tekrarlayalım

$$F_Y(y) = \Pr[Y \le y] = \Pr\left[X > \frac{y-b}{a}\right]$$

$$= 1 - \Pr\left[X \le \frac{y-b}{a}\right] = 1 - F_X\left[\frac{y-b}{a}\right]$$

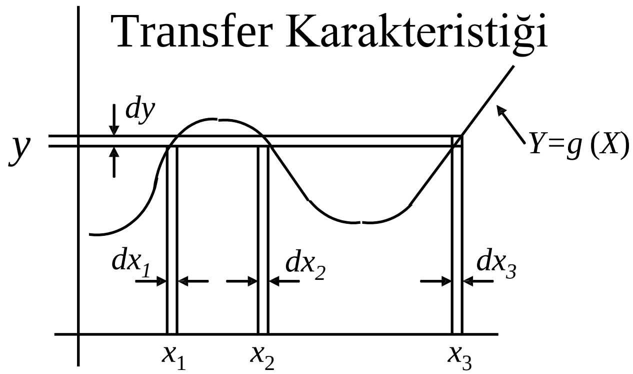


Buradan

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left[1 - F_X\left(\frac{y-b}{a}\right)\right] = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

Rastgele Değişkenin Fonksiyonu

<u>Çok Dğerli Durum</u>

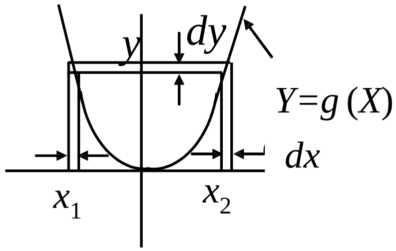


$$\Pr\left[y < Y \le y + dy\right] = \Pr\left[x_1 < X \le x_1 + dx_1\right] + \Pr\left[x_2 < X \le x_2 + dx_2\right]$$
$$+ \ldots + \Pr\left[x_m < X \le x_m + dx_m\right]$$

$$f_Y\left(y\right)\left|dy\right| = f_X\left(x\right)\left|dx_1\right|\Big|_{x=x_1} + f_X\left(x\right)\left|dx_2\right|\Big|_{x=x_2} + \ldots + f_X\left(x\right)\left|dx_m\right|\Big|_{x=x_m}$$

$$f_Y(y) = \sum_{k=1}^{m} \frac{1}{dy/dx} f_X(x)\Big|_{x=g_k^{-1}(y)}$$

Örnek:



$$Y = g(X) = aX^2 \qquad a > 0, -\infty < X < \infty$$

X rastgele değişkeninin oyf si biliniyor. $f_Y(y)$ yi bulun.

$$f_Y(y) = \frac{1}{|dy/dx|} f_X(x)\Big|_{x=g_1^{-1}(y)} + \frac{1}{|dy/dx|} f_X(x)\Big|_{x=g_2^{-1}(y)}$$

$$x_1 = g_1^{-1}(y) = -\sqrt{\frac{y}{a}} \qquad x_1 = g_2^{-1}(y) = +\sqrt{\frac{y}{a}}$$

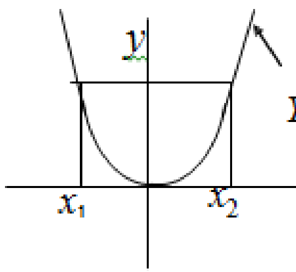
$$\left|\frac{dy}{dx}\right| = |2ax| = 2\sqrt{ay}$$

$$f_Y(y) = \frac{1}{2\sqrt{ay}}\left[f_X\left(-\sqrt{\frac{y}{a}}\right) + f_X\left(+\sqrt{\frac{y}{a}}\right)\right] \qquad y > 0$$

Örnek:

$$Y = g(X) = aX^2 \qquad a > 0$$

r.d. X in oyf si biliniyorsa $f_Y(y)$=?



$$F_Y(y) = \Pr[Y \le y] = \Pr\left[aX^2 \le y\right]$$

$$= \Pr\left[-\sqrt{\frac{y}{a}} \le X \le \sqrt{\frac{y}{a}}\right] = F_X\left(\sqrt{\frac{y}{a}}\right) - F_X\left(-\sqrt{\frac{y}{a}}\right)$$

Buradan

$$f_Y(y) = \frac{d}{dy}\left[F_X\left(\sqrt{\frac{y}{a}}\right) - F_X\left(-\sqrt{\frac{y}{a}}\right)\right] = \frac{1}{2\sqrt{ay}}\left[f_X\left(\sqrt{\frac{y}{a}}\right) - f_X\left(-\sqrt{\frac{y}{a}}\right)\right]$$