Optoelectronics-I

Chapter-4

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Lecture Notes - 2018

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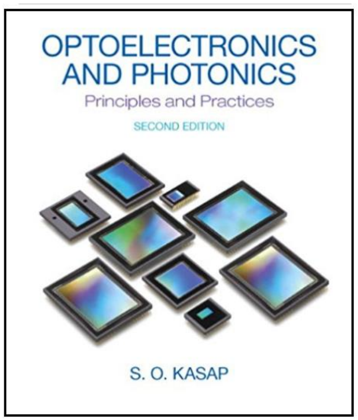
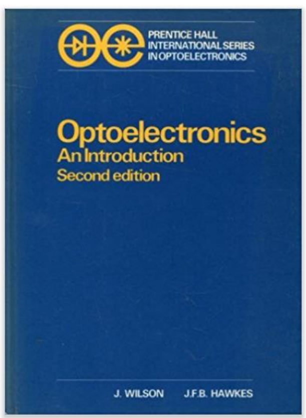
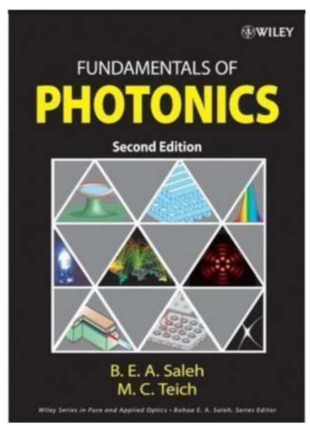
Recommended books







Light Propagation in the Free Space

Objectives

When you finish this lesson you will be able to:

- ✓ Describe the Maxwell Equations in free space
- ✓ To be able to derive classical wave equation from Maxwell's equations
- ✓ Define the electrical and magnetic field components of light
- ✓ Describe the phase and group velocities

Light Propagation in the Free Space

Wave equation

An optical wave is describe mathematically by a real function of position r=(x,y,z) and time t. This function denoted y(r, t) known as the wavefunction.
The wavefunction for a monochromatic wave can be described as,

$$y(r,t) = y_0 \sin\left[\frac{2\pi}{\lambda}(r - \mathrm{v}t)\right]$$

or

$$y(r,t) = y_0 \cos\left[\frac{2\pi}{\lambda}(r - \mathrm{v}t)\right]$$

If the r and t dependency of the function are separated, the function can be rewritten as follows.

$$y(r,t) = y_0(r)cos\left[\frac{2\pi \mathrm{v}}{\lambda}t + \varphi(r)\right] = y_0(r)cos[2\pi \nu t + \varphi(r)]$$

Where, $y_0(r)$ is the position depended amplitude,
v is the wave speed, (remember that c is the speed in the free space)
$\varphi(r)$ is the phase,
ν is the frequency

Light Propagation in the Free Space

Complex Wave equation

It is convenient to represent the real function *y(r,t)* in term of a complex function *Y(r,t)*,

$$Y(r,t) = y(r)\exp([j\varphi(r)]\exp(j2\pi\nu t)$$

So,

$$\mathrm{y}(r,t) = Real\{Y(r,t)\} = \frac{1}{2}[Y(r,t) + Y^*\,(r,t)]$$

The function *Y(r, t)* known as complex wave function. The function *Y(r, t)* satisfy the wave function.

$$\nabla^2 Y - \frac{1}{c^2}\frac{\partial^2 Y}{\partial t^2} = 0$$

The Wave Equation in the free space

$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ ..... Del operator or Lablacian operator

$$Y(r,t) = Y(r)\exp(j2\pi\nu t)$$

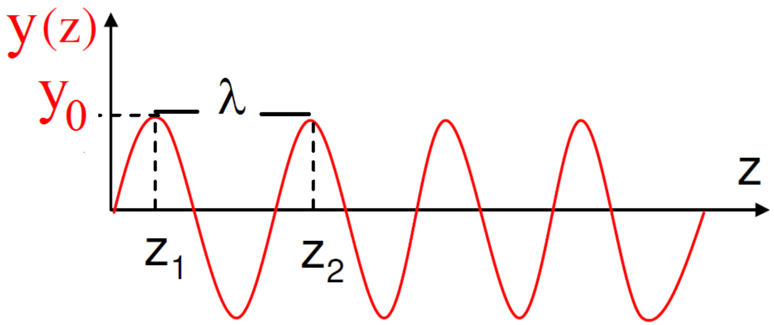
where the time-independent factor $Y(r) = y(r)\exp[\,j\varphi(r)]$ is referred to as the complex amplitude.

Light Propagation in the Free Space

Wave Number

Consider a wave propagating in +z direction.

$$y(r, t) = y_0 \sin\left[\frac{2\pi}{\lambda}(r - vt)\right]$$



$$y(r) = y_0 \sin\left(\frac{2\pi}{\lambda} r\right) = y_0 \sin(kr)$$

At points z1 and z2, a relationship can be established as follows.

$$kz = \frac{2\pi}{\lambda}(z2 - z1) = \frac{5\pi}{2} - \frac{\pi}{2} = 2\pi$$

$$\boldsymbol{k = \frac{2\pi}{\lambda}}$$ **Wave Number**

The wave number k, is the spatial angular frequency of a wave, measured in radians per unit distance. The unit is rad/m.

Light Propagation in the Free Space

Wave Number

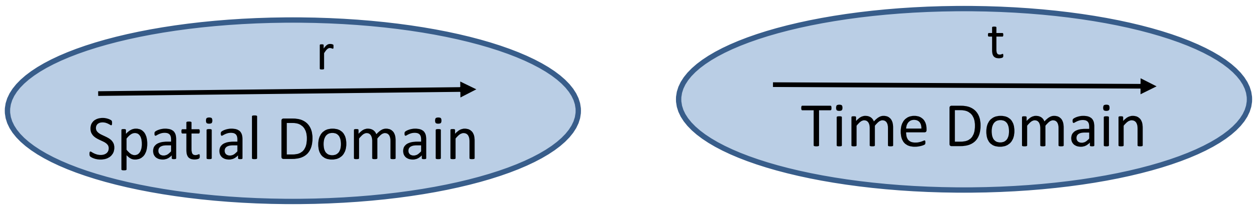
Angular frequency

$y = \sin(kz)$

$$\boldsymbol{k} = \frac{2\pi}{\lambda}$$

$y = \sin(\omega t)$

$$\boldsymbol{\omega} = \frac{2\pi}{T}$$



Scalar wave number

$$\boldsymbol{k} = \frac{2\pi}{\lambda} = \frac{2\pi v}{c}$$

$$\boldsymbol{\omega} = \frac{2\pi}{T} = 2\pi f$$

$\lambda v = c$ in free space

Vector wave number

$$\vec{\boldsymbol{k}} = \frac{2\pi}{\lambda}\hat{\boldsymbol{k}} = \frac{2\pi v}{c}\hat{\boldsymbol{k}}$$

$\hat{k}$ is the unit vector on the wave propagation direction.

Light Propagation in the Free Space

Intensity and Power

The optical intensity ***I(r, t),*** defined as the optical power per unit area (units of watts/cm^2), is proportional to the average of the squared wave function,

$$I(r,t) = 2\langle y^2(r,t)\rangle$$

The operation < … > represents averaging over a time interval that is much longer than the time of an optical cycle.

The optical power ***P(t)*** (units of Watt) flowing in an area A normal to direction of propagation of light is the integrated intensity,

$$P(t) = \int_A I(r,t)dA$$

The ***optical energy*** (units of joules) collected in a given time interval is the time integral of the optical power over the time interval.

Light Propagation in the Free Space

Helmholtz Equation

$$Y(r,t) = y(r)\exp([j\varphi(r)]\exp(j2\pi\nu t)$$

$$Y(r,t) = Y(r)\exp(j2\pi\nu t)$$

$$\boxed{(\nabla^2 + k^2)Y(r)=0}$$

Helmholtz Equation

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c}$$

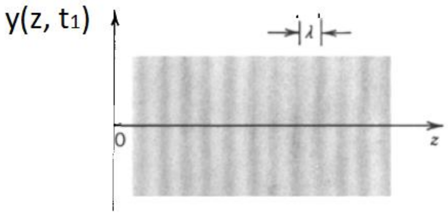
The optical intensity is can be calculated by using the formula given below,

$$I(r) = |Y(r)|^2$$

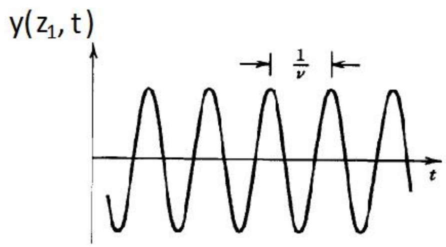
Light Propagation in the Free Space

Plane waves

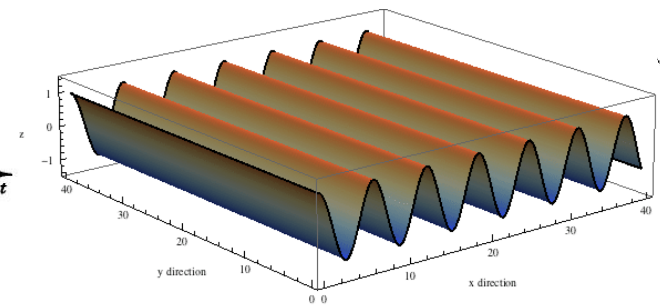
One of the simple solutions of the Helmholtz equation is a plane wave.

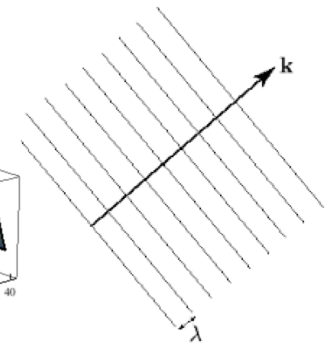


spatial domain



time domain





$$Y = A\exp(-jkz)$$

Amplitude Phase

The plane wave propagates in the +z direction

$$Y = A\exp(-jkr)$$

Amplitude Phase

The plane wave propagates in the +r direction

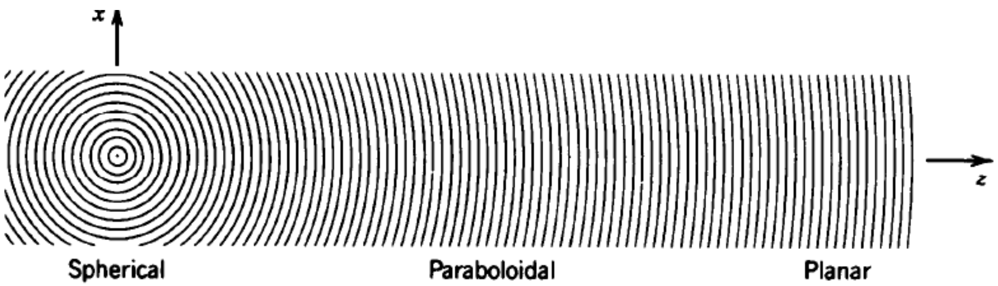
$$Y = A\exp\left[-j(k_x x + k_y y + k_z z)\right]$$

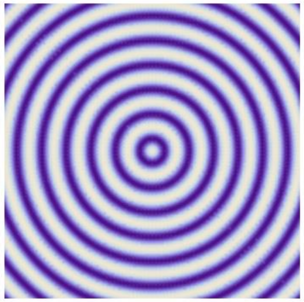
$k = (k_x, k_y, k_z)$ k is the vector wave number

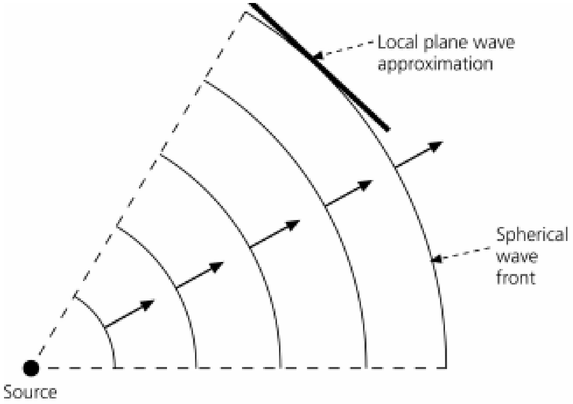
Light Propagation in the Free Space

The Spherical Waves

Another simple solution of the Helmholtz equation is a spherical wave.







$$Y(r) = \underbrace{\frac{A}{r}}_{\text{Amplitude}} \underbrace{\exp(-jkr)}_{\text{Phase}}$$

The spherical wave propagates in **all directions** which decay as 1/r

A spherical wave can be approximated at points near to z axis and sufficiently far from origin by a paraboloidal wave. For very far points, the spherical wave approaches the plane wave.

If a spherical wave is originated at the position r_0,

$$Y(r) = \frac{A}{|r - r_0|}\exp(-jk|r - r_0|)$$

Light Propagation in the Free Space

The Paraboloid Wave

Let us think a spherical wave originating at r = 0 at points r = (x, y, z), the wave is sufficiently close to z axis but far from the origin. So;

$$(x^2 + y^2)^{1/2} \ll z$$

$$\theta^2 = (x^2 + y^2)/z^2 \ll 1$$

$$Y(r) = \frac{A}{r}\exp(-jkr)$$

Taylor series expansion

$$r = (x^2 + y^2 + z^2)^{1/2} = z(1 + \theta^2)^{1/2}$$

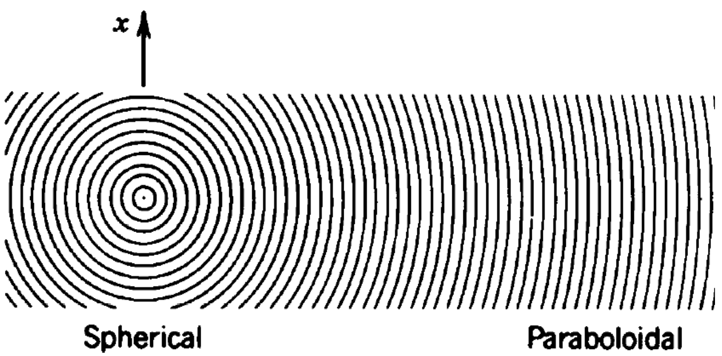
$$= z\left(1 + \frac{\theta^2}{2} - \frac{\theta^4}{8} + \cdots\right)$$

$$\approx z\left(1 + \frac{\theta^2}{2}\right) = z + \frac{x^2 + y^2}{2z}$$

Substituting $r = z + (x^2 + y^2)/2z$

$$Y(\mathbf{r}) \approx \frac{A}{z}\exp(-jkz)\exp\left[-jk\frac{x^2 + y^2}{2z}\right]$$

Fresnel Approximation of a Spherical Wave



Light Propagation in the Free Space

The Paraxial Wave

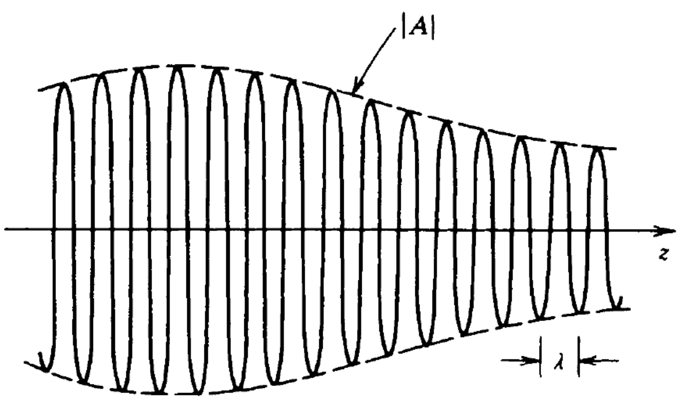
A wave is said to be paraxial if its wavefront normals are paraxial rays.

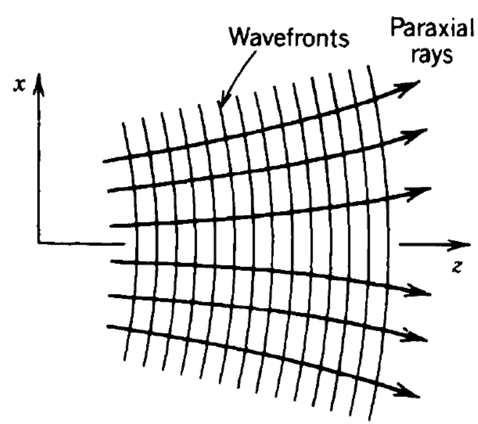
Think a plane wave, $A \exp(-jkz)$,

If the wave have a complex amplitude that similar to slowly varying function such as modulated wave,

$$Y(\mathbf{r}) = A(\mathbf{r}) \exp(-jkz)$$

The variation of $A(\mathbf{r})$ with position must be slow within the distance of a wavelength $\lambda = 2\pi/k$, so that the wave approximately maintains its underlying plane-wave nature.





The paraxial rays travel close to the z axis

Light Propagation in the Free Space

Maxwell Equations

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_o}$$ Gauss' Law for (Electric)

$$\vec{\nabla}.\vec{H} = 0$$ Gauss' Law for Magnetism)

$$\vec{\nabla}\times\vec{E} = -\mu_o \frac{\partial \vec{H}}{\partial t}$$ Faraday's Law)

$$\vec{\nabla}\times\vec{H} = \vec{J}$$ Ampère Law

$$\vec{\nabla}\times\vec{H} = \varepsilon_o \frac{\partial \vec{E}}{\partial t} + \vec{J}$$ Ampère-Maxwell Law

$\vec{E}$ is the electric field vector

H is the magnetic field vector

ρ is the distribution of electric charge

ε_o is the dielectric constant (**permittivity**)

μ_0 is magnetic constant (**permeability**)

for free space

Light Propagation in the Free Space

Maxwell Equations

General form

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_o}$$

$$\vec{\nabla}.\vec{H} = 0$$

$$\vec{\nabla}\times\vec{E} = -\mu_o \frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla}\times\vec{H} = \vec{J}$$

$$\vec{\nabla}\times\vec{H} = \varepsilon_o \frac{\partial \vec{E}}{\partial t} + \vec{J}$$

Medium: Free Space

$$\vec{\nabla}.\vec{E} = 0$$

$$\vec{\nabla}.\vec{H} = 0$$

$$\vec{\nabla}\times\vec{E} = -\mu_o \frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla}\times\vec{H} = \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

Del operator:

$$\vec{\nabla} \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Light Propagation in the Free Space

The Electric and Magnetic Field Equations

$$\vec{E}(x,y,z;t) = E_x(x,y,z;t)\hat{i} + E_y(x,y,z;t)\hat{j} + E_z(x,y,z;t)\hat{k}$$

$$\vec{H}(x,y,z;t) = H_x(x,y,z;t)\hat{i} + H_y(x,y,z;t)\hat{j} + H_z(x,y,z;t)\hat{k}$$

The electric field can be written in the form of a wave equation as follows:

$$\vec{\nabla}\times\vec{E} = -\mu_o \frac{\partial \vec{H}}{\partial t} \qquad \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\mu_o \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H})$$

Any vector A can be written as follows

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = -\nabla^2\vec{A} + \vec{\nabla}.(\vec{\nabla}.\vec{A})$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\mu_o \frac{\partial}{\partial t}\left[\varepsilon_o \frac{\partial \vec{E}}{\partial t}\right]$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\mu_o\varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2} \Rightarrow -\nabla^2\vec{E} + \vec{\nabla}.(\vec{\nabla}.\vec{E}) = -\mu_o\varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2}$$

Light Propagation in the Free Space

The Electric and Magnetic Field Equations

$$-\nabla^2\vec{E}+\vec{\nabla}.(\vec{\nabla}.\vec{E})=-\mu_o\varepsilon_o\frac{\partial^2\vec{E}}{\partial t^2}$$

$$-\nabla^2\vec{E}+\vec{\nabla}.(\vec{\nabla}.\vec{E})=-\mu_o\varepsilon_o\frac{\partial^2\vec{E}}{\partial t^2}$$

We ca write for free space $\vec{\nabla}.\vec{E}=0$

$$\nabla^2\vec{E}=\mu_o\varepsilon_o\frac{\partial^2\vec{E}}{\partial t^2}$$

Electric Field Wave Equation

Remember that

$$\nabla^2 Y-\frac{1}{c^2}\frac{\partial^2 Y}{\partial t^2}=0$$

The Wave Equation in the free space

In that case c must equal to $\frac{1}{\sqrt{\mu_o\varepsilon_o}}=\frac{1}{\sqrt{(4\pi x10^{-7}).(8,85x10^{-13})}}=2,99\times10^8\ m/s$

Light Propagation in the Free Space

The Electric and Magnetic Field Equations

Can you derive the magnetic field wave equation?

Light Propagation in the Free Space

The Electric and Magnetic Field Equations

Maxwell showed that light was an electromagnetic wave.
The light is a transverse electromagnetic wave.